TORSION IN HEEGAARD FLOER HOMOLOGY

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Abstract. We study the Heegaard Floer homology groups of a genus $g$ surface times a circle. We exhibit that their $HF^+$ groups carry 2-torsion (for $g \geq 3$) and 3-torsion (for $g \geq 5$). These are the first known examples to contain any torsion in $HF^\pm$, $\hat{HF}$ or $HF^\infty$.

1. Introduction

Heegaard Floer homology, as introduced by P. Ozsváth and Z. Szabó in [1, 2], assigns to a spin$^c$ 3-manifold $(Y, s)$ a collection of Abelian groups: $HF^\pm(Y, s), \hat{HF}(Y, s), HF^\infty(Y, s)$, of which this article will focus exclusively on $HF^+$. Heegaard Floer homology groups carry additional structure such as an action of $\mathbb{Z}[U] \otimes \Lambda^*(H_1(Y; \mathbb{Z})/\text{Tor})$ and a $\mathbb{Q}$-grading when $c_1(s)$ is torsion. In the latter case we write $HF^+_d(Y, s)$ for the degree $d$ portion of $HF^+(Y, s)$.

Since their inception in 1999, the Heegaard Floer homology groups have been calculated for many classes of 3-manifolds. These include, among others, lens spaces, 3-manifolds obtained by negative definite plumbings, some mapping tori of surfaces, surgeries and double branched covers of knots, etc. Yet, among all these examples there is not one instance whose Heegaard Floer homology groups contain torsion elements. It is therefore natural to ask:

Question Are there any 3-manifolds for which there is torsion in any of the integer-coefficient Heegaard Floer groups?

The purpose of this note is to answer this question in the affirmative. Specifically we prove

Theorem 1. Let $\Sigma_g$ be a surface of genus $g$. Then $HF^+(\Sigma_g \times S^1, s_0)$ contains 2-torsion for all $g \geq 3$ and 3-torsion for all $g \geq 5$. Here $s_0 \in Spin^c(\Sigma_g \times S^1)$ is the unique spin$^c$-structure with $c_1(s_0) = 0$.

2. The Heegaard Floer homology of a surface times a circle

Let us make the identification

$Spin^c(\Sigma_g \times S^1) \cong H^2(\Sigma_g \times S^1; \mathbb{Z}) \cong H^2(\Sigma_g; \mathbb{Z}) \oplus H^1(\Sigma_g; \mathbb{Z})$

With respect to this identification we shall denote by $s_k$ the unique spin$^c$-structure of the form $(\alpha, 0)$ with $\langle c_1(\alpha), [\Sigma_g] \rangle = 2k$. It is easy to see (using
the adjunction inequality) that \( HF^+(\Sigma_g \times S^1, s) = 0 \) unless \( s \) equals some \( s_k \) with \( |k| \leq g - 1 \).

2.1. **The case of** \( k \neq 0 \). The groups \( HF^+(\Sigma_g \times S^1, s_k) \) with \( 0 < |k| \leq g - 1 \) have been calculated by Ozsváth and Szabó [5] for all genera \( g \geq 2 \), and are given by

\[
HF^+(\Sigma_g \times S^1, s_k) \cong H^*(Sym^m(\Sigma_g); \mathbb{Z}) \quad \text{with} \quad m = g - 1 - |k|
\]

In particular they are all free Abelian groups.

2.2. **The case of** \( k = 0 \) **and** \( g \leq 2 \). The groups in these cases are also known by work of Ozsváth and Szabó. As \( \mathbb{Z}[U] \)-modules they are given by \( HF^+(\Sigma_g \times S^1, s_0) \cong \text{HF}(\Sigma_g \times S^1, s_0) \otimes \mathbb{Z}[U, U^{-1}]/U \cdot \mathbb{Z}[U] \) where \( \text{HF}(\Sigma_g \times S^1, s_0) \) are free Abelian groups of total rank 2, 6 and 20 for the genera 0, 1 and 2 respectively [1, 3, 4]. Once again, these groups are torsion free.

2.3. **The case of** \( k = 0 \) **and** \( g \geq 3 \). The groups in this collection were unknown prior to the work of the authors. Our main tool for calculating them is the surgery long exact sequence for Heegaard Floer homology [2]. To explain this, let \( K \) be a nullhomologous knot in a 3-manifold \( Y \) and denote the result of \( n \)-framed surgery along \( K \) by \( Y_n = Y_n(K) \). Then for all sufficiently large \( n \), and with suitable choices of spin\(^c\)-structures (which we suppress from our notation), there is a long exact sequence

\[
(1) \quad \ldots \rightarrow HF^+(Y_n) \xrightarrow{F} HF^+(Y) \rightarrow HF^+(Y_0) \rightarrow HF^+(Y_n) \rightarrow \ldots
\]

Let \( B(0,0) \) be the knot in \((S^1 \times S^2) \# (S^1 \times S^2)\) given by the third component of the Borromean rings after performing 0-framed surgery on the first two components. Taking \( Y = \#^2(S^1 \times S^2) \) and \( K = \#^gB(0,0) \subset Y \), we have \( Y_0 = \Sigma_g \times S^1 \). Thus (1) can be used to find \( HF^+(\Sigma_g \times S^1) \) provided one can get a handle on \( HF^+(Y) \), \( HF^+(Y_n) \) and the map \( F \). All three of these can be computed quite explicitly from the knot Floer homology of \( K \) which has been determined by Ozsváth and Szabó, see [5].

2.3.1. **The \( \mathbb{Z}_2 \)-coefficients case.** The form of the homomorphism \( F \) becomes particularly simple if one uses \( \mathbb{Z}_2 \)-coefficients. Omitting details:

**Theorem 2.** For any \( g \geq 0 \) and for all \( d \) sufficiently large one obtains

\[
(2) \quad \dim_{\mathbb{Z}_2} HF^+_d(\Sigma_g \times S^1, s_0; \mathbb{Z}_2) = 2^{2g-1} + 2^{g-1}
\]

2.3.2. **The \( \mathbb{C} \)-coefficients case.** With \( \mathbb{C} \)-coefficients the map \( F \) from (1) is more intricate. Its kernel turns out to have a form familiar from Kahler geometry. Namely, consider \( H^1(\Sigma_g; \mathbb{C}) \) together with the cup product pairing as a symplectic vector space and let \( e^1, e^2, \ldots, e^{2g-1}, e^{2g} \) be a symplectic basis. Write \( \omega = e^1 \wedge e^2 \pm \ldots \pm e^{2g-1} \wedge e^{2g} \in \Lambda^2 H^1(\Sigma_g; \mathbb{C}) \) for the symplectic form (here we identify \( H^1(\Sigma_g, \mathbb{C}) \) with its dual using the symplectic pairing). Define the **primitive forms** of degree \( j \) to be \( \mathcal{P}^j = \Lambda^j H^1(\Sigma_g, \mathbb{C}) \cap \text{Ker}(i_\omega) \).
where \( \iota_\omega \) denotes contraction with \( \omega \). It is easily checked that \( \dim \mathcal{P}_j = \binom{2g}{j} - \binom{2g-2}{j-2}, \ j = 0, \ldots, g \).

With this notation in place, the kernel of \( F \) in sufficiently high degrees can be identified with the primitive forms \( \mathcal{P}^0 \oplus \mathcal{P}^1 \oplus \cdots \oplus \mathcal{P}^g \) and the cokernel of \( F \) is isomorphic to \( \text{Ker}(F) \) by elementary linear algebra. Putting the calculations together one arrives at

**Theorem 3.** For any \( g \geq 0 \) and all \( d \) sufficiently large one obtains

\[
(3) \quad \dim \mathbb{C} HF^+_d(\Sigma_g \times S^1, s_0; \mathbb{C}) = \binom{2g+1}{g}
\]

Since (3) is strictly smaller than (2) as soon as \( g \geq 3 \), we have that \( HF^+_d(\Sigma_g \times S^1, \mathbb{Z}) \) must contain 2-torsion for all such \( g \). A comparison of (2) and (3) is summarized in the table below.

<table>
<thead>
<tr>
<th>( g )</th>
<th>( \dim \mathbb{Z}_2 HF^+_d(\Sigma_g \times S^1, s_0; \mathbb{Z}_2) )</th>
<th>( \dim \mathbb{C} HF^+_d(\Sigma_g \times S^1, s_0; \mathbb{C}) )</th>
<th>Torsion?</th>
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</tr>
<tr>
<td>( \vdots )</td>
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Using the explicit form of \( F \) in (1), one can calculate the groups \( HF^+_d(\Sigma_g \times S^1, s_0, \mathbb{Z}) \) “by hand” for several low values of \( g \). Doing so for genus 5 one finds that

**Lemma 4.** \( HF^+_d(\Sigma_5 \times S^1, s_0; \mathbb{Z}) \) has elements of order 3 for all sufficiently large values of \( d \).

On the other hand, an inductive argument on \( g \) yields

**Lemma 5.** If \( HF^+(\Sigma_g \times S^1, s_0; \mathbb{Z}) \) contains \( p \)-torsion then so does \( HF^+(\Sigma_{g+1} \times S^1, s_0; \mathbb{Z}) \).

The results from theorem 1 are a combination of the above observations and lemmas 4 and 5.

**References**


