PRETZEL KNOTS, CONCORDANCE AND HEEGAARD FLOER HOMOLOGY

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ABSTRACT. We calculate the Ozsváth-Szabó $\tau$ concordance invariant of the pretzel knots $P(2a+1, 2b+1, 2c+1)$ for any $a, b, c \in \mathbb{Z}$. As an application we re-prove a result first obtained by Fintushel and Stern by which the only pretzel knot with Alexander polynomial 1 which is smoothly slice is the unknot. We also show that most pretzel knots which are algebraically slice are not smoothly slice. A further application shows that the 3-manifolds gotten by $\pm 1$ surgeries on certain pretzel knots are of infinite order in the cobordism group of rational homology 3-spheres.

1. Introduction

Recall that a knot $K$ in $S^3$ is smoothly slice if $(S^3, K) = \partial (B^4, D^2)$ where $D^2 \hookrightarrow B^4$ is a smoothly and properly embedded disk. We say $K_1$ is smoothly concordant to $K_2$ if $K_1 \# (-K_2)$ is smoothly slice. Here $-K$ is $K$ with its orientation reversed and $\overline{K}$ is the mirror of $K$. The smooth concordance relation is an equivalence relation and its equivalence classes under the operation of connected sum form an Abelian group $\mathcal{C}$ – the smooth knot concordance group.

By $P(p, q, r)$ we shall denote the 3-stranded pretzel knot with $p$, $q$ and $r$ half-twists as in figure 1. Pretzel knots satisfy a number of

![Figure 1. The pretzel knot $P(7, -3, 5)$.

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symmetry relations:

\[
P(p, q, r) \cong P(r, p, q) \quad P(p, q, r) \cong P(r, q, p)
\]

\[
P(p, q, r) \cong P(-p, -q, -r)
\]

(1)

The goal of this paper is to calculate the Ozsváth-Szabó concordance invariant \( \tau : \mathcal{C} \to \mathbb{Z} \) (cf. \([10]\)) for certain pretzel knots. Since \( \tau \) of the mirror of a knot is determined by \( \tau \) of the knot (see section 2.4 below), by (1) it suffices to consider pretzel knots \( P(p, q, r) \) with \( p, r \geq 0 \). We shall further restrict our attention to the case where all three \( p, q, r \) are odd and write

\[
p = 2a + 1 \quad q = 2b + 1 \quad r = 2c + 1 \quad a, c \geq 0
\]

The case of pretzel knots with even numbers of half-twists was considered in \([1]\).

**Theorem 1.1.** The value of the Ozsváth-Szabó concordance invariant \( \tau : \mathcal{C} \to \mathbb{Z} \) for 3-stranded pretzel knots with an odd number of twists in each strand is

\[
\tau(P(2a + 1, -2b - 1, 2c + 1)) = \begin{cases} 
0 & \text{if } b \geq \min\{a, c\} \\
-1 & \text{if } b < \min\{a, c\}
\end{cases}
\]

where we assume, without loss of generality, that \( a, c \geq 0 \) (see (1)).

Recall that when \( K \) is an alternating knot then \( \tau(K) = -\sigma(K)/2 \) (cf. \([10]\)) where \( \sigma(K) \) is the signature of \( K \). When \( b < 0 \) in theorem 1.1 the knot \( P(2a + 1, -2b - 1, 2c + 1) \) is alternating so the result of theorem 1.1 in that case follows from a signature calculation:

\[
\sigma(P(2a+1, -2b-1, 2c+1)) = \begin{cases} 
0 & ; \quad b > \frac{4ac-1}{4(a+c+1)} \\
2 & ; \quad b < \frac{4ac-1}{4(a+c+1)}
\end{cases} \quad \text{with } a, c \geq 0.
\]

Theorem 1.1 has a number of interesting consequences. To illustrate these recall that the Alexander polynomial of \( P(p, q, r) \) is given by

\[
\Delta_{P(p,q,r)}(t) = \frac{1}{4}((pq + pr + qr)(t - 2 + t^{-1}) + (t + 2 + t^{-1}))
\]

showing that a pretzel knot \( P(p, q, r) \) has trivial Alexander polynomial precisely when

\[
pq + pr + qr = -1
\]

By work of Freedman \([4]\) all knots with trivial Alexander polynomial are topologically slice. In contrast we have
Corollary 1.2. Among the pretzel knots $P(p,q,r)$ with trivial Alexander polynomial and with $p,q,r$ odd only the unknot is of finite order in $C$.

This result was first obtained by Fintushel and Stern [3] by means of Donaldson gauge theory. Their approach was to consider the double covers $Y(p,q,r)$ of $S^3$ branched over $P(p,q,r)$. When equation (4) is satisfied $Y(p,q,r)$ is just the Brieskorn sphere $\Sigma(|p|,|q|,|r|)$. Fintushel and Stern showed that no connected sum $\#_k \Sigma(|p|,|q|,|r|)$, $k\geq 1$ can bound a $\mathbb{Z}_2$-acyclic 4-manifold. To date there is no proof of corollary 1.2 in the context of Seiberg-Witten theory.

To state our next application recall first that if $K$ is a slice knot then its Alexander polynomial $\Delta_K(t)$ factors as $\Delta_K(t) = f(t) \cdot f(t^{-1})$ for some polynomial $f(t)$ with integer coefficients. For pretzel knots $P(p,q,r)$ with $p,q,r$ odd this condition is equivalent to

$$pq + pr + qr = -n^2$$

for some odd integer $n$ (in which case $f(t) = \frac{1-n}{2} t + \frac{1+n}{2}$). In fact, it was shown by Levine [5] that condition (5) is equivalent to $P(p,q,r)$ being algebraically slice.

Corollary 1.3. Consider pretzel knots $P(p,q,r)$ with $p,q,r$ odd and with $pq + qr + pr = -n^2$ for some odd integer $n$. Without loss of generality (see (1)) assume that $p \geq r \geq 1$. If $P(p,q,r)$ is of finite order in $C$ then $q$ and $r$ are subject to the inequalities

$$0 < r \leq n \quad -n \leq q \leq -r$$

For a fixed $n$ and with the additional assumption that $q + r \neq 0$, there are at most $(n^2 - 1)/8$ pretzel knots $P(p,q,r)$ with finite order in $C$.

In the case when $q + r = 0$ and $P(p,q,r)$ is of finite order, corollary 1.3 implies that the knot has the form $P(n+k,-n,n)$ for some $k \geq 0$ (up to symmetry (1)). All such knots have both $\sigma$ and $\tau$ equal to zero and there may well be infinitely many finite order knots among these.

Many families of pretzel knots satisfying the hypotheses of corollary 1.3 exist, for example for a fixed $n \in \mathbb{N}$ consider (cf. [2])

$$P(n(4k+3),-n(2k+1),n(4k+1)) \quad ; \quad k \in \mathbb{N}$$
$$P(n(2k^2+4k+1),-n(2k+1),n(2k+3)) \quad ; \quad k \in \mathbb{N}$$
$$P(n(k^2+3k+1),-n(2k+1),n(2k+5)) \quad ; \quad k \in \mathbb{N}$$
$$P(n(12k+5),-n(4k+1),n(6k+3)) \quad ; \quad k \in \mathbb{N}$$
$$P(n(12k+7),-n(4k+3),n(6k+5)) \quad ; \quad k \in \mathbb{N}$$

(6)

The results of theorem 1.1 further underscore the difference between $\tau$ and the invariants $\delta$ and $-\sigma/2$. Here $\delta$ is the invariant defined by
Manolescu and Owens [6]. Recall that all three of these invariants agree for alternating knots [10, 6]. On the other hand, consider any of the families of knots from (6) with $n = 1$. Each of these knots has trivial Alexander polynomial and so their double branched covers are integral homology spheres. As such, their correction terms are integers and $\delta$, which is defined as twice the correction term of the unique spin-structure on the double branched covers, must be even. For example $\delta(P(7, -3, 5)) = 4$, cf. [11]. However $\tau$ of any of the knots from (6) equals $-1$ (for any value of $n \geq 1$) according to theorem 1.1.

To compare $\tau$ to $-\sigma/2$, consider pretzel knots $P(p, q, r)$ (with $p, q$ still odd and $p > 0$). Then theorem 1.1 and equation (2) show that $\tau$ and $-\sigma/2$ differ on the interval $-p < q < -p/2$ which can be made arbitrarily large.

However, as the calculations from [19] show, $\tau$ and the Rasmussen knot concordance invariant $s$ (cf. [18]) are in perfect agreement for pretzel knots $P(p, q, r)$ with $p, q, r$ odd.

To state our next result we first recall some more definitions. The integral homology cobordism group $\Theta^3_\mathbb{Z}$ is the group generated by equivalence classes $[Y]_\mathbb{Z}$ of integral homology 3-spheres $Y$ under the operation of taking connected sums. The equivalence relation $\sim$ inducing $[\cdot]_\mathbb{Z}$ is given by

$$Y_1 \sim Y_2 \text{ if there exists an integral homology 4-ball } X \text{ with } \partial X = -Y_1 \# Y_2,$$

where $-Y$ is $Y$ with the opposite orientation. The rational homology cobordism group $\Theta^3_\mathbb{Q}$ is generated by equivalence classes $[(Y, s)]_\mathbb{Q}$ consisting of a rational homology sphere $Y$ and a spin\(^c\)-structure $s$ on $Y$. The operation on $\Theta^3_\mathbb{Q}$ is again that of taking connected sums while the equivalence relation $\sim$ inducing the classes $[\cdot]_\mathbb{Q}$ is now

$$(Y_1, s_1) \sim (Y_2, s_2) \text{ if there exists a } \text{spin}\(^c\) \text{-homology 4-ball } (X, t) \text{ with } \partial X = -Y_1 \# Y_2, \text{ and } t|_{\partial X} = \overline{s}_1 \# s_2$$

where $\overline{s}$ is the conjugate spin\(^c\)-structure of $s$. Let us denote the map $[Y]_\mathbb{Z} \mapsto [(Y, s_0)]_\mathbb{Q}$ (where $s_0$ is the unique spin-structure on the integral homology sphere $Y$) by $\mathcal{Z} : \Theta^3_\mathbb{Z} \to \Theta^3_\mathbb{Q}$.

Both groups $\Theta^3_\mathbb{Z}$ and $\Theta^3_\mathbb{Q}$ are rather poorly understood. One of the main tools in the study of $\Theta^3_\mathbb{Z}$ has been the Rohlin invariant $\mu$ which induces an epimorphism $\mu : \Theta^3_\mathbb{Z} \to \mathbb{Z}_2$. The Ozsváth-Szabó correction term $d(Y, s)$ (see [8] and section 2.2 below) provides a homomorphism $d : \Theta^3_\mathbb{Z} \to \mathbb{Q}$. It is known that $\mathcal{Z} : \Theta^3_\mathbb{Z} \to \Theta^3_\mathbb{Q}$ is neither injective nor surjective.
Theorem 1.4. Consider a pretzel knot $P(p,q,r)$ with $p, q, r$ odd and let $Y_\ell$ be the 3-manifold obtained by $\ell$-framed surgery on $P(p,q,r)$. Then if $\tau(P(p,q,r)) = -1$ the manifold $Y_{-1}$ is of infinite order in $\Theta^3_Z$ while if $\tau(P(p,q,r)) = 1$ then $Y_1$ is of infinite order in $\Theta^3_Z$. In each case the images under $\mathbb{Z}$ of these manifolds remain of infinite order in $\Theta^3_Q$.

None of the manifolds $Y_{\pm 1}$ from the above theorem are Seifert fibered spaces (see [17] for a Heegaard Floer proof of this fact by Ozsváth and Szabó). As such they are not easily accessible with the existing tools for the study of $\Theta^3_Z$. The Rohlin invariant of the manifolds $Y_{\pm 1}$ is
\[
\mu(Y_{\pm 1}) \equiv \frac{pq + qr + pr + 1}{4} \pmod{2}
\]
which vanishes in many instances covered by theorem 1.4. Specifically it vanishes for all knots from (6) with any choice of $n$. When $P(p,q,r)$ is alternating, the conclusions of theorem 1.4 follow from a calculation of the correction term $d(Y_{\pm 1})$ by Ozsváth and Szabó (cf. [9]). In fact, for alternating $P(p,q,r)$ with $\tau(P(p,q,r)) = -1$ according to [9] one gets $d(Y_{-1}) = -2$ and when $\tau(P(p,q,r)) = 1$ one obtains $d(Y_1) = 2$.

The proof of theorem 1.4 relies on theorem 1.1.

The remainder of the article is organized as follows. In section 2 we review the relevant notions from Heegaard Floer homology needed in later sections. An effort is made to make the article as self-contained as possible and the reader is only referred the foundational articles [15, 14] by Ozsváth and Szabó in regards to technical details which don’t affect the flow of the discussion. Section 3 is devoted to the proof of theorem 1.1 and its corollaries. The final section ?? provides a proof of theorem 1.4 in a slightly more general setting.

Acknowledgements I would like to thank Swatee Naik for sharing her expertise on knot concordance and Nik Saveliev for a helpful email exchange.

2. Heegaard Floer homology

2.1. Heegaard Floer homology groups. In [15, 14] Ozsváth and Szabó introduced the Heegaard Floer homology groups
\[
\overline{HF}(Y) \quad HF^\pm(Y) \quad HF^\infty(Y)
\]
associated to a 3-manifold $Y$. These groups are the homologies of chain complexes $(\overline{CF}(Y), \partial), (CF^\pm(Y), \partial^\pm), (CF^\infty(Y), \partial^\infty)$ associated to a pointed Heegaard diagram $(\Sigma_g, \{\alpha_1, \ldots, \alpha_g\}, \{\beta_1, \ldots, \beta_g\}, z)$ of $Y$. The generators over $\mathbb{Z}$ of $CF^\infty(Y)$ are of the form $[x, i]$ with $i \in \mathbb{Z}$ and $x = \{x_1, \ldots, x_g\}$ with $x_\ell \in \alpha_\ell \cap \beta_{\sigma(\ell)}$ for some permutation $\sigma$ on $g$ letters.
The differential $\partial^\infty$ of a generator $[x, i]$ is a linear combination of terms $[y, j]$ with $j \leq i$. This implies that the subgroup $CF^-(Y)$ of $CF^\infty(Y)$ generated by those $[x, i]$ with $i < 0$ is a subcomplex of $CF^\infty(Y)$. $CF^+(Y)$ is defined as the quotient complex $CF^\infty(Y)/CF^-(Y)$. Each of the complexes $CF^\pm(Y)$ comes equipped with an action of $\mathbb{Z}[U]$ defined on the generators by $U[x, i] = [x, i - 1]$. The complex $\hat{CF}(Y)$ is the kernel of $U : CF^+(Y) \to CF^+(Y)$. The differentials $\partial^\pm$ and $\hat{\partial}$ are the obvious ones induced by $\partial^\infty$.

Each of the complexes $CF^0(Y)$ is a direct sum of complexes $CF^\circ(Y, s)$ with $s \in Spin^c(Y)$. We omit the details and refer the interested reader to [15, 14]. In this article, with few exceptions, $Y$ will be $S^3$ and thus $s$ will be immaterial.

The Heegaard Floer groups possess a host of additional algebraic structures of which we point out some for later use:

- There is a long exact homology sequence

\[(\text{7}) \quad \ldots \to HF^{-}(Y, s) \to HF^\infty(Y, s) \xrightarrow{\partial} HF^{+}(Y, s) \to \ldots\]

associated to the short exact sequence of complexes
\[0 \to CF^{-}(Y, s) \to CF^\infty(Y, s) \to CF^{+}(Y, s) \to 0\]

- When $s$ is torsion (by which we mean that $c_1(s) \in H^2(Y; \mathbb{Z})$ is torsion) then each of $HF^\circ(Y, s)$ comes equipped with a rational grading $\tilde{gr} : HF^\circ(Y, s) \to \mathbb{Q}$. We shall write $\hat{HF}(d)(Y, s)$ for the portion of $\hat{HF}(Y, s)$ in grading $d$ and use similar notation for the other groups.

- The action of $U$ lowers the grading by 2: $\tilde{gr}(U \cdot x) = \tilde{gr}(x) - 2$ for each homogeneous generator $x$.

**Theorem 2.1** (Ozsváth-Szabó, [13]). Let $Y$ be a 3-manifold and $K \subset Y$ a nullhomologous knot. Denote by $Y_\ell$ the 3-manifold obtained by $\ell$-framed surgery on $K$. Then for any $n > 0$ there is a long exact sequence

\[(\text{8}) \quad \ldots \to \hat{HFK}(Y) \to \hat{HFK}(Y_n) \to \hat{HFK}(Y_0) \to \hat{HFK}(Y) \to \ldots\]

2.2. **The correction term.** The correction term $d(Y, s) \in \mathbb{Q}$ (cf. Ozsváth-Szabó [8]) is defined for any torsion spin$^c$-structure $s$ on $Y$ as

\[d(Y, s) = \min \{ \tilde{gr}(\pi(x)) \mid x \in HF^\infty(Y, s) \}\]

1The precise formula of the differential will be immaterial for our discussion and is thus suppressed. The interested reader can consider [15, 14] for full details.
with $\pi$ and $\overline{g}r$ as in section 2.1. The correction term satisfies the equations (proved in [8])

$$
\begin{align*}
    d(Y, \overline{s}) &= d(Y, s) \\
    d(-Y, s) &= -d(Y, s) \\
    d(Y_1 \# Y_2, s_1 \# s_2) &= d(Y_1, s_1) + d(Y_2, s_2)
\end{align*}
$$

and descends to a homomorphism $d : \Theta^3_Q \rightarrow \mathbb{Q}$.

2.3. Knot Floer homology. A nullhomologous knot $K$ in a 3-manifold $Y$ gives rise to a chain complex $(CFK^\infty(Y, K), \partial^\infty)^2$ generated over $\mathbb{Z}$ by triples $[x, i, j]$ with $i, j \in \mathbb{Z}$ and with $x = \{x_1, \ldots, x_\theta\}$ as in section 2.1.

The differential of a generator $[x, i, j]$ is a linear combination of terms $[y, i', j']$ with $i' \leq i$ and $j' \leq j$. This gives rise to a number of subcomplexes and quotient complexes of $CFK^\infty(Y, K)$ which for reasons of simplicity of notation we shall denote by

$$
C\{\text{conditions on } i, j\}
$$

thus dropping reference to both $Y$ and $K$. In the above the “conditions on $i, j$” will typically be given in terms of inequalities whose nature determines whether the complex is a subcomplex, a quotient complex or indeed a quotient complex of a subcomplex. For example

$$
\begin{align*}
    C\{i < 0\} &= \text{Subcomplex generated by } [x, i, j], i < 0 \\
    C\{i \geq 0\} &= \text{Quotient complex } CFK^\infty(Y, K)/C\{i < 0\} \\
    C\{i < 0 \text{ or } j < 0\} &= \text{Subcomplex generated by } [x, i, j], i < 0 \text{ or } j < 0 \\
    C\{i \geq 0 \text{ and } j \geq 0\} &= \text{Quotient complex } CFK^\infty(Y, K)/C\{i < 0 \text{ or } j < 0\}
\end{align*}
$$

As in the case of the Heegaard Floer homology groups from section 2.1, there is an action of $\mathbb{Z}[U]$ on $CFK^\infty(Y, K)$ also (and its various associated complexes) given by $U [x, i, j] = [x, i - 1, j - 1]$.

In analogy with the 3-manifold case, the knot Floer group $HF^\infty(Y, K)$ is a direct sum of groups $HF^\infty(Y, K, t)$ with $t$ ranging over Spinc$(Y_0)$ (see [13] for complete details). Here $Y_0$ is the 3-manifold obtained from $Y$ by 0-framed surgery on $K$. Upon selecting a Seifert surface $F \subset Y$ for $K$ there is a bijective correspondence between Spinc$(Y_0)$ and Spinc$(Y) \times \mathbb{Z}$ provided by

$$
(10) \quad t \mapsto \left( s, \frac{1}{2} \langle c_1(t), [\hat{F}] \rangle \right)
$$

\footnote{By abuse of notation the two differentials occurring in sections 2.1 and 2.3 are both denoted by $\partial^\infty$ even though they are not the same.}
where \( s \in \text{Spin}^c(Y) \) is the unique spin\(^c\)-structure on \( Y \) with \( t|_{Y_0-N(K)} = s|_{Y-N(K)} \) (with \( N(K) \) being a tubular neighborhood of \( K \)) and where \( \hat{F} \subset Y_0 \) is the surface obtained from \( F \) by attaching the meridional disk of \( K \) to \( F \). Using this correspondence we will write \( CFK_{\infty}(Y,K,s,j) \) to mean \( CFK_{\infty}(Y,K,t) \) if \( t \mapsto (s,j) \) under the above correspondence. The spin\(^c\)-structure \( s \) is often left out when clear from context. This decomposition of \( CFK_{\infty}(Y,K) \) by spin\(^c\)-structure descends to the various sub- and quotient-complexes associated to it.

We single out a particularly important complex derived from \( CFK_{\infty}(Y,K,t) \), namely the complex \( C_{\{i=0\}} \) (which is the kernel of \( U : C_{\{i \geq 0\}} \to C_{\{i \geq 0\}} \)). This complex can be thought of as a filtered version of \( \hat{CF}(Y,s) \) (with \( t \) and \( s \) related as in (10)) associated to the filtration map \( \mathfrak{F}_K : \hat{CF}(Y,s) \to \mathbb{Z} \) given by

\[
\mathfrak{F}_K([x,0]) = \frac{1}{2} \langle c_1(t), [\hat{F}] \rangle
\]

The filtration of \( \hat{CF}(Y,s) \) induced by \( \mathfrak{F}_K \) is

\[
\mathcal{F}_K(\ell) = \text{Subcomplex of } C_{\{i=0\}} \text{ generated by } [x,0,j] \text{ with } j \leq \ell
\]

where \( j = \mathfrak{F}_K([x,0]) \). We define \( \hat{CFK}(Y,K,s,j) \) to be the associated graded object of this filtration

\[
\hat{CFK}(Y,K,s,j) = \frac{\mathcal{F}_K(j)}{\mathcal{F}_K(j-1)}
\]

and denote its homology by \( \hat{HF}(Y,K,s,j) \). Here too we will often suppress \( s \) from the notation.

As in the case of the Heegaard Floer groups, the various knot Floer groups carry additional structure. For example, when \( s \) (not \( t \)) is torsion then the groups \( \hat{HF}(Y,K,s,j) \) and \( \hat{HF}(Y,K,s,j) \) come equipped with a rational grading \( \tilde{gr} \). We will again indicate the portion of \( \hat{HF}(Y,K,s,j) \) in grading \( d \) by a subscript \( \hat{HF}_d(Y,K,s,j) \). We will refer to \( j \) as the filtration level of \( \hat{HF}_d(Y,K,s,j) \).

2.4. The \( \tau \) invariant. The filtration \( \mathcal{F}_K \) on \( \hat{CF}(Y,s) \) from section 2.3 gives rise to a convergent spectral sequence \( E^r_K \) (see for example [7]) with

\[
E^2_K \cong \hat{HF}(Y,K,t) \quad \text{and} \quad E^\infty_K \cong \hat{HF}(Y,s)
\]

(with \( t \) and \( s \) related again as in (10)). In fact \( E^r_K \) abuts to \( E^\infty_K \) after finitely many levels \( r \).

In [10] Ozsváth and Szabó defined an integral invariant \( \tau(K) \) of a knot \( K \) in \( S^3 \) by taking advantage of the filtration \( \mathcal{F}_K \). To see how,
recall first that $\widehat{H\mathcal{F}}(S^3) \cong \mathbb{Z}_{(0)}$. Moreover, for a knot $K$ in $S^3$ let us write $\widehat{H\mathcal{F}}(K,j)$ to mean $\widehat{H\mathcal{F}}(S^3, K, s_0, j)$ for the unique spin-structure $s_0$ on $S^3$. Following [10] we make the definition

$$\tau(K) = \min \left\{ j \in \mathbb{Z} \mid \iota_* : H_*(\mathcal{F}_K(j)) \to \widehat{H\mathcal{F}}(S^3) \text{ is surjective.} \right\}$$

where $\iota : \mathcal{F}_K(j) \to CF(S^3)$ is the inclusion map. By the comment on spectral sequences at the beginning of this subsection if follows that the set on the right-hand side above is not empty.

We summarize the relevant properties of $\tau(K)$ in the next theorem.

Recall that the smooth slice genus $g_*(K)$ of a knot $K$ in $S^3$ is the smallest genus of any smoothly and properly embedded surface $\Sigma \hookrightarrow B^4$ with $\partial(B^4, \Sigma) = (S^3, K)$.

**Theorem 2.2** (Ozsváth - Szabó, [10]). For an oriented knot $K$ in $S^3$ let again $-K$ and $\overline{K}$ be the reverse and mirror of $K$ respectively. Then

$$\tau(K) = \tau(-K) = -\tau(K)$$

Moreover $\tau$ induces a surjective homomorphism $\tau : C \to \mathbb{Z}$ on the smooth knot concordance group and provides a lower bound for the smooth slice genus of $K$

$$|\tau(K)| \leq g_*(K)$$

**2.5. The skein sequence.** In [13] Ozsváth and Szabó proved that if $K_+$ and $K_-$ are two knots in $S^3$ which only differ in a single crossing (at which they look as in figure 2) and $K_0$ is the 2 component link obtained by resolving that crossing (see again figure 2) then there is a long exact sequence relating their $\widehat{H\mathcal{F}}$ groups:

$$\ldots \to \widehat{H\mathcal{F}}_d(K_-, j) \to \widehat{H\mathcal{F}}_{d-1}(K_0, j) \to \widehat{H\mathcal{F}}_{d-1}(K_+, j) \to \ldots$$

This skein sequence is a filtered version of the sequence (8) for $Y = S^3$

**Figure 2.** The knots $K_+$ and $K_-$ and the 2-component link $K_0$. associated to $\infty$, $-1$ and 0-surgery along the knot $\gamma$ from figure 3.
With these choices the three 3-manifolds from (8) are $$S^3 = S^3_\infty(\gamma)$$,
$$S^3 = S^3_{-1}(\gamma)$$ and $$S^1 \times S^2 = S^3_0(\gamma)$$ turning (8) into

$$\gamma \cong \gamma$$

**Figure 3.** -1 surgery on $$\gamma$$ changes a positive crossing
to a negative crossing.

$$(12)$$

$$(\ldots \to \hat{HFK}_{(d)}(S^3_{-1}(\gamma)) \to \hat{HFK}_{(d-\frac{1}{2})}(S^3_0(\gamma)) \to \hat{HFK}_{(d-1)}(S^3_\infty(\gamma)) \to \ldots)$$

The sequences (11) and (12) are related as elucidated in the next lemma.

**Lemma 2.3.** Let $$K_+$$, $$K_-$$ and $$K_0$$ be as above (see figure 2) and assume
that the Seifert genera of $$K_+$$ and $$K_-$$ are both equal to $$g$$. Then there
is a commutative diagram with exact rows

$$(\ldots \to \hat{HFK}_{(d)}(K_+, -g) \to \hat{HFK}_{(d)}(K_-, -g) \to \hat{HFK}_{(d-\frac{1}{2})}(K_0, -g) \to \ldots)$$

**Proof.** It was proved by Ozsváth and Szabó in [12] that for any knot $$L$$
in $$S^3$$ the knot Floer homology $$\hat{HFK}(L, j)$$ is zero when $$|j| > g(L)$$ and
nonzero for $$|j| = g$$. Since by assumption the genera of $$K_+$$ and $$K_-$$ are
both equal to $$g$$, it follows that $$\hat{HFK}(K_\pm, j) = 0$$ for all $$j < -g$$. The
exact sequence (11) shows that then $$\hat{HFK}(K_0, j) = 0$$ for $$j < -g$$ also.

Let $$L$$ be any of $$K_\pm$$ or $$K_0$$, then the short exact sequence

$$0 \to \mathcal{F}_L(-g - 1) \to \mathcal{F}_L(-g) \to \mathcal{F}_L(-g) / \mathcal{F}_L(-g - 1) \to 0$$

induces an isomorphism on homology

$$H_*(\mathcal{F}_L(-g)) \cong \hat{HFK}(L, -g)$$

Consider then the maps $$\hat{HFK}(L, -g) \cong H_*(\mathcal{F}_L(-g)) \to \hat{HF}(Y)$$ (where
$$Y = S^3$$ if $$L = K_\pm$$ and $$Y = S^1 \times S^2$$ when $$L = K_0$$) induced by the
inclusion map $$\mathcal{F}_L(-g) \to \mathcal{CF}(Y)$$. Since the maps from (11) are merely
filtered versions of the maps from (12) (see proposition 8.1 from [13]) and since $\widehat{HFK}(L, j) = 0$ for all $j < -g$, the claim of the lemma follows. □

2.6. The conjugation map. The definition of the various knot Floer homology groups from section 2.3 depends on an orientation of $K$, however, their isomorphism type does not. A change of orientation from $K$ to $-K$ is on the level of chain complexes reflected by the existence of a chain homotopy equivalence which we shall refer to as the conjugation map (and which is canonical up to sign cf. [16])

$$J : CFK^\infty(Y, K, t) \to CFK^\infty(Y, -K, t)$$

The algebraic consequence of reversing the orientation of $K$ is that the indices $i$ and $j$ in the generators $[x, i, j]$ of $CFK^\infty(Y, K, t)$ “trade places”. This is not meant literally (it is certainly not true that $J([x, i, j]) = [x, j, i]$) but rather implies that the map $J$ descends to maps (all still denoted $J$)

$$J : C\{\text{Condition on } (i, j)\} \to C\{\text{Condition on } (j, i)\}$$

For example there are conjugation maps

$$J : C\{i = 0\} \to C\{j = 0\} \quad \text{and} \quad J : C\{i \geq 0\} \to C\{j \geq 0\}$$

All the various conjugation maps are $U$-equivariant.

Lemma 2.4. Let $K$ be a genus $g$ knot in $S^3$ and let $J : C\{i \geq 0\} \to C\{j \geq 0\}$ be the conjugation map as above. Then generators of the form $[x, 0, -g]$ are under $J$ mapped to generators of the form $[y, -g, 0]$ or $[y, 0, g]$.

Proof. Consider first the conjugation map $J : C\{i + j \leq \ell\} \to C\{i + j \leq \ell\}$ where $\ell$ is any integer. Since $J$ is $U$-equivariant it has to map $C\{i + j = \ell\}$ to itself (as elements from $C\{i + j = \ell\}$ are the only terms of $C\{i + j \leq \ell\}$ which are not in the image of $U$).

Using this observation the claim of the lemma follows from the fact that $C\{i + j = -g\} \cap C\{i \geq 0\}$ is generated by terms of the form $[x, 0, -g]$ and likewise $C\{i + j = -g\} \cap C\{j \geq 0\}$ is generated by terms of the shape $[y, -g, 0]$. □

2.7. Relation of knot Floer homology to Heegaard Floer homology. This section explains how the Heegaard Floer homology groups of the 3-manifolds $Y_\ell = Y_\ell(K)$ obtained by $\ell$-framed surgery on a null-homologous knot $K \subset Y$ are related to the knot Floer homology of $K$. We only concern ourselves with the case of $Y = S^3$, the case of an arbitrary $Y$ can be found in [13]. We first explain some notation.
Let $W_\ell$ be the cobordism from $S^3$ to $S^3_\ell$ obtained by attaching an $\ell$-framed 2-handle to $S^3 \times [0,1]$ along $K \times \{1\}$. Let $s_k$ be the spin$^c$-structure on $S^3_\ell$ which is the restriction of the unique spin$^c$-structure $t_k$ on $W_\ell$ which obeys the condition $\langle c_1(t_k), [S] \rangle = 2k - n$ where $S$ is the generator of $H_3(W_\ell; \mathbb{Z})$ obtained by capping of the Seifert surface $F$ of $K$ by the core of the attaching 2-handle. With this understood we have

**Theorem 2.5** (Ozsváth-Szabó, [13]). Let $K$ be a genus $g$ knot in $S^3$. Then for any $n \geq 2g - 1$ the Heegaard Floer groups $HF^+(S^3)$ and $HF^+(S^3_{3-n}, s_0)$ are isomorphic to

\[
HF^+(S^3) \cong H_4(C\{i \geq 0\}) \quad HF^+(S^3_{3-n}, s_0) \cong H_4(C\{i \geq 0 \text{ and } j \geq 0\})
\]

Moreover, in the long exact sequence (8)

\[
... \rightarrow HF^+(S^3) \xrightarrow{F} HF^+(S^3_{3-n}, s_0) \rightarrow HF^+(s^3_0, s_0) \rightarrow ...
\]

the map $F$ is the map in homology induced by the chain map $f : C\{i \geq 0\} \rightarrow C\{i \geq 0 \text{ and } j \geq 0\}$ given by

\[
f = \pi_+ + \pi_- \circ J + \text{lower degree terms}
\]

with $\pi_+$ being projection onto $C\{i \geq 0 \text{ and } j \geq 0\}$ and $J : C\{i \geq 0\} \rightarrow C\{j \geq 0\}$ being the conjugation map from section 2.6. The “lower order terms” are in reference to the rational grading $\tilde{gr}$ from section 2.1.

**2.8. Cobordism induced maps.** A spin$^c$-cobordism $(W, t)$ from a spin$^c$ 3-manifold $(Y_0, s_0)$ to a spin$^c$ 3-manifold $(Y_1, s_1)$ is a 4-manifold $W$ with boundary $\partial W = -Y_0 \sqcup Y_1$ along with a spin$^c$-structure $t$ on $W$ with $t|_{Y_i} = s_i$. Ozsváth and Szabó (cf. [15, 14]) show that a spin$^c$ cobordism $(W, t)$ induces maps

\[
F^\circ_{W,t} : HF^\circ(Y_0, s_0) \rightarrow HF^\circ(Y_1, s_1)
\]

with $\circ$ being any of $+, -, \infty, \sim$. These maps are compatible with the long exact sequence (7) in that there is a commutative diagram with exact rows

\[
\begin{array}{cccccc}
\cdots & HF^{-}(Y_0, s_0) & \xrightarrow{F_{W,t}^-} & HF^\circ(Y_0, s_0) & \xrightarrow{\pi} & HF^{+}(Y_0, s_0) & \cdots \\
\downarrow & F_{W,t}^- & & F_{W,t}^\circ & & F_{W,t}^+ \\
\cdots & HF^{-}(Y_0, s_0) & \xrightarrow{F_{W,t}^-} & HF^\circ(Y_0, s_0) & \xrightarrow{\pi} & HF^{+}(Y_0, s_0) & \cdots
\end{array}
\]

When both $s_0$ and $s_1$ are torsion (and thus both groups $HF^\circ(Y_i, s_i)$ are equipped with a rational grading $\tilde{gr} : HF^\circ(Y_i, s_i) \rightarrow \mathbb{Q}$) the degree
of $F_{W,t}^\circ$ is

$$\deg F_{W,t}^\circ = \frac{c_1(t)^2 - 2e_W - 3\sigma_W}{4}$$  \hspace{1cm} (15)$$

where $e_W$ and $\sigma_W$ are the Euler number and signature of $W$.

Observe that in the special case when $W$ is obtained by attaching a $-1$-framed 2-handle to a knot $K \times \{1\}$ in $S^3 \times [0,1]$ the degree shift (15) becomes

$$\deg F_{W,t_k}^\circ = \frac{1 - (2k + 1)^2}{4}$$

where $t_k$ is the spin$^c$-structure on $W$ characterized by $\langle c_1(t_k), [S] \rangle = -1 - 2k$ and $S$ is the surface obtained from the Seifert surface $F$ of $K$ by capping it off with the core 2-disk of the attaching 2-handle.

The map $F : HF^+(S^3) \to HF^+(S^3_{-1}, s_0)$ from (13) is of the form $F = \sum_{k \in \mathbb{Z}} F_{W,t_k}^+$ and can be written as the sum

$$F = F_{W,t_0}^+ + F_{W,t_1}^+ + \sum_{k \neq 0, -1} F_{W,t_k}^+$$  \hspace{1cm} (16)$$

in which the first two summands on the right-hand side have degree zero and the remaining summands have negative degree. This decomposition is the same as the decomposition from (14) in which the “lower order terms” correspond to the map $\sum_{k \neq 0, -1} F_{W,t_k}^+$.

**Theorem 2.6** (Ozsváth-Szabó, [8]). Let $W$ be a cobordism between the rational homology 3-spheres $Y_0$ and $Y_1$ and assume that $b_2^+ (W) = 0$. Then $F_{W,t}^\infty : HF^\infty(Y_0, s_0) \to HF^\infty(Y_1, s_1)$ is an isomorphism for each $t \in \text{Spin}^c(W)$.

### 3. The $\tau$ invariant of pretzel knots

Once again, let $P(p, q, r)$ denote the 3-stranded pretzel knot with $p$, $q$ and $r$ half-twists. Recall the symmetries mentioned in the introduction

$$P(p, q, r) \cong P(q, r, p) \quad P(p, q, r) \cong P(r, q, p) \quad P(p, q, r) \cong P(-p, -q, -r)$$

where as before $\overline{K}$ is the mirror of $K$. As $\tau$ of a mirror $\overline{K}$ is determined by $\tau$ of the knot (cf. theorem 2.2), we shall restrict our considerations to pretzel knots of the form $P(p, q, r)$ with $p, r \geq 1$. Recall also that we are working with the assumption that all three integers $p, q, r$ are odd numbers. For simplicity of notation later on let us write

$$p = 2a + 1 \quad q = \pm (2b + 1) \quad r = 2c + 1 \quad a, b, c \geq 0$$

Since the Seifert genus of all such pretzel knots is 0 or 1, by theorem 2.2 it follows that $|\tau(P(p, q, r))| \leq 1$. 


When \( p = 2a + 1, q = 2b + 1 \) and \( r = 2c + 1 \) are all positive the corresponding pretzel knots are alternating and so their \( \tau \)-invariant equals \(-\sigma/2\) where the signature \( \sigma \) is as in (2). This together with the next theorem yields theorem 1.1.

**Theorem 3.1.** The \( \tau \)-invariant for the pretzel knots \( P(2a + 1, -2b -1, 2c + 1) \) with \( a, b, c \geq 0 \) is given by

\[
\tau(P(2a + 1, -2b -1, 2c + 1)) = \begin{cases} 
0 &; \ b \geq \min\{a, c\} \\
-1 &; \ b < \min\{a, c\}
\end{cases}
\]

**Proof.** The claim of the theorem in the case of \( b \geq \min\{a, c\} \) follows from the calculation of \( \hat{\text{HFK}}(P(2a + 1, -2b -1, 2c + 1), \pm 1) \) by Ozsváth and Szabó from \[
\hat{\text{HFK}}(P(2a + 1, -2b -1, 2c + 1), j) \cong \begin{cases} 
Z_{1}^{ab+bc+b-ac} &; \ j = 1 \\
Z_{-1}^{ab+bc+b-ac} &; \ j = -1
\end{cases}
\]

Since the filtration levels \( j = \pm 1 \) do not contain any elements of grading zero, \( \tau \) of these knots must equal zero.

The case of \( b < \min\{a, c\} \) is proved similarly using as input a further result from \[
\hat{\text{HFK}}(P(2a + 1, -2b -1, 2c + 1), j) \cong \begin{cases} 
Z_{1}^{(b+1)} \oplus Z_{(2)}^{-(a-b)(c-b)} &; \ j = 1 \\
Z_{-1}^{(b+1)} \oplus Z_{(0)}^{-(a-b)(c-b)} &; \ j = -1
\end{cases}
\]

We next invoke the skein relation (11) and the results of lemma 2.3. Let \( K_{-} = P(2a + 1, -2b -1, 2c + 1) \) and pick the crossing from the skein relation (11) to be one of the crossings in the third column of half-twists (the one with \( 2c+1 \) half-twists). Then \( K_{+} = P(2a + 1, -2b -1, 2c - 1) \) and \( K_{0} = T_{2,2(a-b)} \), the \((2, 2(a-b))\) torus link. If we think of \( T_{2,2(a-b)} \) as sitting in the interior of a solid torus, then the strands of \( T_{2,2(a-b)} \) are oriented so as to have linking number zero with the meridian of the torus. With this orientation \( T_{2,2(a-b)} \) has signature 1 and Alexander polynomial equal to

\[
\Delta_{T_{2,2(a-b)}}(t) = -(a-b)(t^{1/2} - t^{-1/2})
\]
Since \( T_{2,2(a-b)} \) is alternating, these two determine its Floer homology completely [9]:

\[
\widehat{HF}(T_{2,2(a-b)}, j) = \left\{ \begin{array}{ll}
\mathbb{Z}^{a-b} & j = 1 \\
\mathbb{Z}^{2(a-b)} & j = 0 \\
\mathbb{Z}^{a-b} & j = -1
\end{array} \right.
\]

When \( c = b + 1 \) the knot \( K_+ \) falls into the \( b \geq \min\{a, c\} \) category and so we treat this case separately. Thus when \( c = b + 1 \) the skein sequence (11) with \( j = -1 \) and lemma 2.3 yield the commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{Z}^{a-b}_0 & \rightarrow & \mathbb{Z}^{a-b}_{(-\frac{1}{2})} & \rightarrow & \mathbb{Z}^{b(b+1)}_{(-1)} & \rightarrow & \mathbb{Z}^{b(b+1)}_{(-1)} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & \mathbb{Z}^{a-b}_0 & \rightarrow & \mathbb{Z}^{a-b}_{(-\frac{1}{2})} & \rightarrow & \mathbb{Z}^{b(b+1)}_{(-1)} & \rightarrow & \mathbb{Z}^{b(b+1)}_{(-1)} & \rightarrow & 0
\end{array}
\]

The above shows that the horizontal maps

\[
\mathbb{Z}^{a-b}_0 \rightarrow \mathbb{Z}^{a-b}_{(-\frac{1}{2})} \quad \text{and} \quad \mathbb{Z}^{b(b+1)}_{(-1)} \rightarrow \mathbb{Z}^{b(b+1)}_{(-1)}
\]

are isomorphisms. On the other hand (17) implies that the vertical map \( \mathbb{Z}^{a-b}_{(-\frac{1}{2})} \rightarrow \mathbb{Z}^{(-\frac{1}{2})} \) is surjective. Thus we conclude that the vertical map \( \mathbb{Z}^{a-b}_0 \rightarrow \mathbb{Z}_0 \) is also surjective. As this latter map is the map

\[
\widehat{HF}(\mathcal{F}_{K_+}(-1)) \rightarrow \widehat{HF}(S^3)
\]

it follows that \( \tau(P(2a+1, -2b-1, 2b+1)) = -1 \).

When \( c \geq b + 2 \) then both \( K_- \) and \( K_+ \) fall into the range of \( b < \min\{a, c\} \) and the analogue of the above commutative diagram now looks like

\[
\begin{array}{ccccccccc}
\mathbb{Z}^{(a-b)(c-a-1)}_0 & \rightarrow & \mathbb{Z}^{(a-b)(c-b)}_0 & \rightarrow & \mathbb{Z}^{a-b}_{(-\frac{1}{2})} & \rightarrow & \mathbb{Z}^{b(b+1)}_{(-1)} & \rightarrow & \mathbb{Z}^{b(b+1)}_{(-1)} & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
& & \mathbb{Z}^{(a-b)(c-a-1)}_0 & \rightarrow & \mathbb{Z}^{(a-b)(c-b)}_0 & \rightarrow & \mathbb{Z}^{a-b}_{(-\frac{1}{2})} & \rightarrow & \mathbb{Z}^{b(b+1)}_{(-1)} & \rightarrow & \mathbb{Z}^{b(b+1)}_{(-1)} & \rightarrow & 0
\end{array}
\]

The same line of argument as in the case of \( c = b + 1 \) again implies that \( \tau(K_-) = -1 \) completing the proof of the theorem. \( \square \)

Proof of corollary 1.3. Since any knot \( P(p, q, r) \) with finite order in \( C \) has \( \tau \) equal to zero, theorem 1.1 implies \( q \leq -r \) (by our assumption \( 0 < r \leq p \)). Then the equation \( pq + pr + qr = -n^2 \) implies that
\[-n^2 \leq -r(p + r) + pr = -r^2\] which gives \(0 < r \leq n\). Using this once more in \(pq + pr + qr = -n^2\) gives
\[-n^2 \geq n(p + q) + pq = q(p + n) + np \implies -n \leq q \leq -r\]
Each choice of \(q, r\) with \(q + r \neq 0\) determines \(p\) uniquely as
\[p = \frac{-n^2 - qr}{q + r}\]

\textit{Proof of corollary 1.2.} Assume that \(P(p, q, r)\) is of finite order in \(C\) and that \(p, q, r\) are odd and \(0 < r \leq p\). Corollary 1.3 \((n = 1)\) then shows that \(r = 1\) and \(q = -1\). But \(P(p, -1, 1)\) is the unknot for any value of \(p\). □

4. ±1 surgeries on pretzel knots

This section is devoted to the proof of theorem 1.4. Given a knot \(K\) in \(S^3\) let us again \(Y_\ell\) to denote the 3-manifold obtained from \(K\) by \(\ell\)-framed surgery. As the proof below shows theorem 1.4 holds true for any knot \(K\) of genus 1 with \(\tau(K) = \pm 1\).

\textit{Proof of theorem 1.4.} Let \(K\) be a knot in \(S^3\) with Seifert genus equal to 1 and with \(\tau(K) = -1\). Let \(W\) be the cobordism from \(S^3\) to \(Y_{-1}\) obtained by attaching a \(-1\)-framed 2-handle to \(K \times \{1\}\) in \(S^3 \times [0, 1]\). Consider the commutative diagram (see section 2.8)

\[
\begin{array}{ccc}
HF_{-}^\infty(S^3) & \xrightarrow{F^\infty_W} & HF_{-}^\infty(Y_{-1}) \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
HF_{+}^+(S^3) & \xrightarrow{F^+_W} & HF_{+}^+(Y_{-1})
\end{array}
\]

Since \(b_2^+(W) = 0\) it follows from theorem 2.6 that \(F^\infty_W\) is an isomorphism. Let \(y \in HF_{-}^\infty(Y_{-1})\) be an element with \(\tilde{g}_W(\pi_2(y)) = d(Y_{-1})\). Let \(x \in HF_{-}^\infty(S^3)\) be the preimage of \(y\) under \(F^\infty_W\). Since \(\tilde{g}_W(\pi_1(x)) \geq 0\) and since \(\deg F^+_W = 0\) (see (15)) it follows that \(0 \leq \tilde{g}_W(\pi_1(x)) = \tilde{g}_W(\pi_2(y)) = d(Y_{-1})\).

Suppose that in fact \(d(Y_{-1}) = 0\). This assumption implies the \(F^+_W : HF_{+}^+(S^3) \to HF_{+}^+(Y_{-1})\) is injective which, as we shall see presently, contradicts formula (14) from theorem 2.5 along with the hypothesis \(\tau(K) = -1\). Namely, by lemma 2.4 all generators of \(CF_{+}^+(S^3) \cong C\{i \geq 0\}\) of the form \([x, 0, -1]\) lie in the kernel of \(f = \pi_+ + \pi_- \circ J\). But since \(\tau(K) = -1\) one of the generators of this type must survive to homology thus giving a nontrivial kernel of \(F^+_W\), a contradiction. We are therefore forced to conclude that \(d(Y_{-1}) > 0\).
When $K$ is a genus 1 knot with $\tau(K) > 0$ then
\[
d(Y_1(K)) = -d(-Y_1(K)) = -d(Y_{-1}(K)) > 0
\]
This last inequality follows from $\tau(K) = -\tau(\overline{K})$ (theorem 2.2) and the first half of the proof.

\section*{References}


