The antiderivative $\int e^{x^2} \, dx$ cannot be expressed in terms of so-called "elementary functions". Therefore, you can't find $\int_0^1 e^{x^2} \, dx$ exactly, like you can $\int_0^1 x^2 \, dx = 1/3$. It must be approximated.

Of course you can use Riemann sums (like The Left-hand Riemann sum; Right-hand Riemann sum; or Midpoint Riemann sum) to approximate an integral. However these are usually not efficient. You have to use lots of subintervals to get a good approximation.

Efficiency is an important practical issue in approximating complicated integrals. If an integral is very complex it may require a great deal of computer time, which can be expensive. Therefore finding an efficient way to get an answer is important.

In section 7.7 we will discuss two new ways of approximating an integral $\int_a^b f(x) \, dx$. They are the **Trapezoidal Rule** and **Simpson's Rule**.
The **Trapezoidal Rule** and **Simpson's Rule** still depend on doing what we did with Riemann sums. Namely, we partition the interval \([a, b]\) into subintervals of equal length \(\Delta x\). Then we label *equally spaced* points \(x_i\) in the interval:

\[
a = x_0 < x_1 < x_2 \ldots < x_{n-1} < x_n = b, \quad x_i - x_{i-1} = \Delta x
\]

1. The left-hand Riemann sum of the function \(f(x)\) on the interval \([a, b]\) is \(\sum_{i=0}^{n-1} f(x_i) \Delta x\) which can be written as \(\Delta x \sum_{i=0}^{n-1} f(x_i)\).

2. The right-hand Riemann sum of the function \(f(x)\) on the interval \([a, b]\) is \(\Delta x \sum_{i=1}^{n} f(x_i)\).

3. The midpoint Riemann sum of the function \(f(x)\) on the interval \([a, b]\) is \(\Delta x \sum_{i=1}^{n} f(\bar{x}_i)\), where \(\bar{x}_i\) is the midpoint of the subinterval \([x_{i-1}, x_i]\).

These three Riemann sums are sums of areas of rectangles.

4. The trapezoidal rule is a sum of areas of trapezoids. It is also the average of the right-hand and left-hand Riemann sums. It can be written as

\[
\frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)].
\]
5. Simpson’s rule is a sum of areas of parabolic regions that are a much closer approximation in general than the other four above. It requires an even number of subintervals.

2 subintervals:
\[
\Delta x \frac{1}{3} [f(x_0) + 4f(x_1) + f(x_2)].
\]

4 subintervals:
\[
\Delta x \frac{1}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)].
\]

6 subintervals:
\[
\Delta x \frac{1}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + f(x_6)].
\]

n subintervals (n must be even):
\[
\Delta x \frac{1}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \ldots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].
\]
\texttt{evalf(\texttt{ApproximateInt}(f(x), x=a..b, method=left, partition=n)));\texttt{ApproximateInt}(f(x), x=a..b, method=left, partition=n, output=plot)}; 

1.222594957
\textbf{Problem 3.8}

\begin{verbatim}
> evalf(ApproximateInt(f(x), x=a..b,
    method=right, partition=n)); ApproximateInt(f(x), x=a..b,
    method=right, partition=n, output=plot);
\end{verbatim}

\begin{align*}
0.9362199488
\end{align*}

\begin{center}
\includegraphics[width=\textwidth]{plot.png}
\end{center}
> evalf(ApproximateInt(f(x), x=a..b, method=midpoint, partition=n));
> ApproximateInt(f(x), x=a..b, method=midpoint, partition=n, output=plot);

1.146968562
\[ \frac{\Delta x}{2} \left[ f(0) + 2f(\frac{1}{2}) + 2f(1) + 2f(\frac{3}{2}) + f(2) \right] \]

\[ \Delta x = \frac{1}{2} \]
\[
\Delta x = \frac{1}{2}, \quad n = 4 \text{ sub-intervals}
\]
> evalf(ApproximateInt(f(x), x=a..b, method=simpson, partition=n));
> ApproximateInt(f(x), x=a..b, method=simpson, partition=n, output=plot);

### NOTE: Maple uses 2n subintervals!

\[
\sum_{i=1}^{n} f(x_i)
\]

1.124448193
Approximate $\int_a^b f(x) \,dx, \quad x=a..b,$
method=right, partition=n, output=plot;

An Approximation of the Integral of
$f(x) = x^2$
on the Interval [1, 3]
Using a Right-endpoint Riemann Sum

Area: 10.75000000

$$\Delta x \sum_{i=1}^{n} f(x_i) = \frac{1}{2} \left( 1.5^2 + 2^2 + 2.5^2 + 3^2 \right)$$
$$= 10.75$$
\[ f(x) = x^2 \text{ on } [1, 3] \]

\[ \text{with(Student[Calculus1])}: \quad 4 \text{ subintervals} \]

\[ f := x \rightarrow x^2; a := 1; b := 3; n := 4; \# \ n = \text{Number of partition subintervals} \]

\[ \text{ApproximateInt}(f(x), x = a..b, \text{method=left, partition=n, output=plot}); \]

An Approximation of the Integral of \( f(x) = x^2 \)

on the Interval \([1, 3]\)

Using a Left-endpoint Riemann Sum

Area: 6.750000000

\[ \sum_{i=0}^{3} f(x_i) \]

\[ \Delta x \sum_{i=0}^{3} f(x_i) = \frac{1}{2} \left[ 1^2 + (1.5)^2 + 2^2 + (2.5)^2 \right] \]

\[ = 6.75 \]
> ApproximateInt(f(x), x=a..b, method=midpoint, partition=n, output=plot);

An Approximation of the Integral of

\[ f(x) = x^2 \]

on the Interval [1, 3]

Using a Midpoint Riemann Sum

Area: 8.625000000

\[
\Delta x \sum_{b=1}^{4} f(\bar{x}_b) = \\
\frac{1}{2} \left[ 1.25^2 + 1.75^2 + 2.25^2 + 2.75^2 \right] = 8.625
\]
An Approximation of the Integral of
\[ f(x) = x^2 \]
on the Interval [1, 3]
Using the Trapezoid Rule

\[ \text{Area: 8.750000000} \]

\[ \Delta x = \frac{3 - 1}{4} = 0.5 \]

\[ \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4) \right] \]
\[ = \frac{1}{4} \left[ 1^2 + 2(1.5)^2 + 2(2^2) + 2(2.5)^2 + 3^2 \right] \]
\[ = 8.75 \]
\[ (= \text{Average of Left and Right Sums}) \]
An Approximation of the Integral of
f(x) = x^2
on the Interval [1, 3]
Using Simpson's Rule

Area: 8.666666667

See next page
Simpson with 4 subintervals

\[ \frac{4}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right] \]

\[ \Delta x = \frac{1}{2} \]

\[ \frac{1}{6} \left[ 1^2 + 4(1.5)^2 + 2(2)^2 + 4(2.5)^2 + 3^2 \right] \]

\[ = \frac{1}{6} \left[ 1 + 9 + 8 + 25 + 9 \right] \]

\[ = \frac{52}{6} \]

\[ = 8 \frac{2}{3} \]

\[ = 8.666 \ldots \]

Note \[ \int_{1}^{3} x^2 \, dx = \frac{x^3}{3} \bigg|_{1}^{3} = 9 - \frac{1}{3} = 8 \frac{2}{3} \]
Example: Estimate \( \int_{1}^{3} \frac{1}{x} \, dx \) using (1) the Trapezoidal rule and (2) Simpson's rule both with 4 subintervals.

(1)

\[
\Delta x \left[ \frac{1}{2} \left( f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3) + f(x_4) \right) \right]
\]

\[
\frac{1}{4} \left[ 1 + 2\left( \frac{2}{3} \right) + 2\left( \frac{1}{2} \right) + 2\left( \frac{2}{3} \right) + \frac{1}{3} \right]
\]

\[
\frac{1}{4} \left[ 1 + \frac{4}{3} + 1 + \frac{4}{5} + \frac{1}{3} \right]
\]

\[
= \frac{67}{60} \approx 1.117
\]
Simpson's Rule

\[ \Delta x \left[ \frac{1}{3} f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4) \right] \]

\[ \frac{1}{6} \left[ f(1) + 4f(\frac{3}{2}) + 2f(2) + 4f(\frac{5}{2}) + f(3) \right] \]

\[ \frac{1}{6} \left[ 1 + \frac{8}{3} + 1 + \frac{8}{5} + \frac{1}{3} \right] \]

= \frac{11}{10} = 1.1

Actual \[ \int_1^3 \frac{1}{x} \, dx = \ln 3 - \ln 1 \]

= \ln 3

= 1.0986122

1.117
Example:
The widths (in meters) of a kidney-shaped swimming pool were measured at two-meter intervals as indicated in the figure. Estimate the area of the pool using (a) Simpson's rule and (b) the trapezoidal rule.

Area = \int_a^b W(x) \, dx = \int_0^{12} W(x) \, dx.

Applying Simpson's rule with \( \Delta x = 2 \) and \( n = 6 \) gives this:

\[
\frac{\Delta x}{3} [W(x_0) + 4W(x_1) + 2W(x_2) + 4W(x_3) + 2W(x_4) + 4W(x_5) + W(x_6)]
\]

\[
= \frac{2}{3} [7.2 + 4 \times 6.8 + 2 \times 5.6 + 4 \times 5 + 2 \times 4.8 + 4 \times 4.8 + 0]
\]

\[
= \frac{2}{3} \times 94.4 = 62.95 \text{ square meters.}
\]
Now using the trapezoidal rule, we get:

\[
\frac{\Delta x}{2}[W(x_0) + 2W(x_1) + 2W(x_2) + 2W(x_3) + 2W(x_4) + 2W(x_5) + W(x_6)]
\]

\[
7.2 + 2 \times 6.8 + 2 \times 5.6 + 2 \times 5 + 2 \times 4.8 + 2 \times 4.8 + 0
\]

\[
= 40.8 \text{ square meters.}
\]
Suppose we want to approximate the integral \( \int_{a}^{b} f(x) \, dx \) using the trapezoidal rule or Simpson's rule with \( n \) subintervals in the partition of the interval \([a, b]\). How accurate are the approximations?

Let \( K \) be a number such that \( |f''(x)| \leq K \) for all values of \( x \) in the interval \([a, b]\). The maximum error in using the trapezoidal rule to approximate this integral does not exceed \( \frac{K(b - a)^3}{12n^2} \).

Let \( K \) be a number such that \( |f^{(4)}(x)| \leq K \) for all values of \( x \) in the interval \([a, b]\). The maximum error in using Simpson's rule to approximate this integral does not exceed \( \frac{K(b - a)^5}{180n^4} \).
Example: Suppose \(-6 \leq f''(x) \leq 4\) on \([1, 3]\). What is the smallest number of subintervals necessary to ensure that the trapezoidal rule approximates \(\int_1^3 f(x) \, dx\) to within \(10^{-5}\)?

In this example, we can take \(K = 6\). Since \(b - a = 2\), our error formula tells us that we want

\[
10^{-5} \geq \frac{6 \cdot 2^3}{12n^2}.
\]

If we solve this inequality for \(n^2\) we obtain

\[
n^2 \geq \frac{6 \cdot 2^3 \cdot 10^5}{12} = 400,000.
\]

Therefore, \(n \geq \sqrt{400,000} \approx 632.5\), which means that \(n\) should be at least 633 because \(n\) must be a whole number. So the final answer to this example is \(n = 633\).
**Example:** Suppose \(-1 \leq f^{(4)}(x) \leq 4\) on \([1, 6]\). What is the smallest number of subintervals necessary to ensure that Simpson’s rule approximates \(\int_{1}^{6} f(x) \, dx\) to within \(10^{-5}\)?

In this example, we can take \(K = 4\). Since \(b - a = 5\), our error formula tells us that we want

\[
10^{-5} \geq \frac{4 \times 5^5}{180n^4}.
\]

If we solve this inequality for \(n^4\) we obtain

\[
n^4 \geq \frac{4 \times 5^5 \times 10^5}{180} = 6944444.
\]

Therefore, \(n \geq 6944444^{1/4} \approx 51.3\), which means that \(n\) should be at least 52 because \(n\) must be a whole number. So the final answer to this example is \(n = 52\).

Also note that \(n\) must be an even number in Simpson’s rule, therefore if the inequality had come out saying \(n \geq \approx 52.3\), we would have to say that the value of \(n\) we seek is not 53, but 54.