NOTES ON LOGIC AND PROOFS

In mathematics a statement is a sentence that is either true or false (but not both).

Which of these is a statement in the above sense? (a) \(25 = 1\) (b) \(x^2 = 4\) (c) \(3x^2 + x\) (d) \(7 \geq 1\) (e) \(\sqrt{2}\) (f) “This sentence is false.”

The truth of a statement like “he is short” is ambiguous. Some people might call a 5 foot 7 inch man short while others would say that he’s not short, he’s just sort of medium. Such ambiguity is not acceptable in mathematics. If a statement is used in mathematics or logic we must agree that it is true or false but not both.

There are lots of unsolved problems in mathematics, i.e., statements that are either true or false, but we don’t know which. A statement like “the universe is infinite” is one that scientists just don’t know the answer to, but it’s assumed to be either true or false.

Keeping in mind that statements are assumed to be either true or false, is this statement true? “All the elephants in my pocket are wearing ballet slippers.”

A proof is an argument that uses logical principles to conclude the truth of a statement from facts that are known or assumed true.

A theorem is a statement for which we have a proof. In mathematical writing, the term “theorem” is typically reserved for true statements whose proofs are not short or obvious. Other names for theorems, depending on the circumstances, are lemma, proposition, and corollary.

A conjecture is an assertion that we suspect may be true, but has not been proved.

Formal logic is a vital tool for mathematics. Thus, we must understand how the following logical terms are used: “not”, “and”, “or”, “if ... then”, and “if and only if”.

OR and AND

There is no question as to what “and” means, but this is not the case with “or”.

In ordinary conversation, if I say “I will take you to the 3 o’clock showing of Harry Potter or the 7 o’clock showing,” I don’t mean I’ll take you to both. What I mean is one or the other but not both. This is an illustration of the so-called “exclusive or.” On the other hand, if a waitress says, “Would you like butter or sour cream on your potato?” she means one or the other or possibly both. This is an illustration of the so-called “inclusive or.”

In mathematics we always use the word “or” in the inclusive sense – one or the other or possibly both. Thus all three of the following statements are true
(2 + 2 = 4) OR (3 + 3 = 5), (2 + 2 = 5) OR (3 + 3 = 6), (2 + 2 = 4) OR (3 + 3 = 6).

The only “or” statement that is false is one with both component parts false, as for example (2 + 2 = 5) OR (3 + 3 = 5).

\textbf{NOT}

The \textit{negation} of a statement \(p\) is sometimes written \(\sim p\) and it means “it is not the case that \(p\)”. \(\sim p\) is true when, and only when \(p\) is false.

\textbf{Exercise 1} The negation of \(p\ OR q\) is \(\sim (p\ OR q)\). Write this in another way.

\(\sim p\ AND\ \sim q\).

\textbf{Exercise 2} Write the negation of this: \((2 + 2 = 4)\ OR\ (3 + 3 < 6)\) without using \(\sim\) or the word “not”.

\((2 + 2 \neq 4)\ AND\ (3 + 3 \geq 6)\)

\textbf{Exercise 3} The negation of \(p\ AND\ q\) is \(\sim (p\ AND\ q)\). Write this in another way.

\(\sim p\ OR\ \sim q\).

\textbf{Exercise 4} Write the negation of this: \((2 + 2 = 4)\ AND\ (3 + 3 < 6)\) without using \(\sim\) or the word “not”.

\((2 + 2 \neq 4)\ OR\ (3 + 3 \geq 6)\)

\textbf{Exercise 5} Write the negation of this: “All the elephants in my pocket are wearing ballet slippers.”

"There is an elephant in my pocket not wearing ballet slippers."

\textbf{PARENTHESES.}

What does this mean? \(3 < 1\ AND\ 2 = 2\ OR\ 5 = 5\). Does it mean \((3 < 1\ AND\ 2 = 2)\ OR\ 5 = 5\) or does it mean \(3 < 1\ AND\ (2 = 1\ OR\ 5 = 5)\)? The first is true but the second is false.

The point is that parentheses are important. You must use them to avoid ambiguity.

Another example is \(\sim p\ AND\ q\). It could mean \(\sim (p\ AND\ q)\) or \((\sim p)\ AND\ q\). These are not the same. Usually \(\sim p\ AND\ q\) is interpreted as \((\sim p)\ AND\ q\) just like \(-2 + 3\) means \((-2) + 3 = 1\) and not \((-2 + 3) = -5\). However, using parentheses ensures no confusion.
IF ... THEN ...

A sentence of the form “if \( p \) then \( q \)” is called a conditional. The sentence following the word “if” is called the **hypothesis** or the **antecedent**. The part following “then” is called the **conclusion** or **consequent**.

Few words are so misused and misunderstood as “if ... then ...”, but it is absolutely vital that you understand clearly the meaning of these words as we will use them.

Think of a statement of the form “if \( p \) then \( q \)”, as a promise with a condition (in fact, this is why such statements are called conditionals). I promise that if \( p \) happens then \( q \) will happen too. The promised \( q \) need not be fulfilled unless the condition \( p \) is met.

If you think about what it takes to make this statement false, there’s only one way - that’s when \( p \) is true but \( q \) is false. Otherwise you really can’t say the promise has been broken. And if you can’t say it’s false in the other cases, then there’s only one other choice - it’s got to be true, because every statement in mathematics is either true or false.

Suppose I promise that you will get an “A” in the course if you have an average which is 90% or greater. If I am telling you the truth, then you will certainly expect an “A” if your average is 92%. But if your average is 89%, I can give you an “A” anyway without breaking my promise. I could also give you a “B” without breaking my promise. The question of what will occur for an average that is less than 90% is simply not addressed by the promise as stated, using those special words “if” and “then”.

You may use the notation \( p \Rightarrow q \) to stand for “if \( p \) then \( q \)”, but you must use it correctly. One often reads \( p \Rightarrow q \) as “\( p \) implies \( q \)”.

**Exercise 6** Is it correct to write “if \( x = 5 \) then \( x + 2 = 7 \)” as “if \( (x = 5) \Rightarrow (x + 2 = 7) \)”?

NO. Just “\( (x = 5) \Rightarrow (x + 2 = 7) \)” without the ”if”.

**Exercise 7** Which of these are correct? (a) \( 2 + 2 \Rightarrow 4 \). (b) \( (x + 2 = 3) = (x = 1) \).
(c) \( (x + 2 = 3) \Rightarrow (x = 1) \).

Only (c) is correct.

**Exercise 8** Which of these are true? (1) If \( 2 + 2 = 5 \) then \( 1 \leq 2 \). (2) If \( 2 + 2 = 5 \) then 3 is an even number.

Both are true.

**Exercise 9** I am going to prove that \( 1 = -1 \). Here it is: \( 1 = -1 \Rightarrow 1^2 = (-1)^2 \Rightarrow 1 = 1 \), which is true. This ends my proof! What’s wrong?
The argument is starting with a false statement. From that any conclusion can follow.

**Exercise 10** I am going to prove that \( x = -2 \) and \( x = 1 \) are solutions to the equation \( \frac{x^2 + x - 2}{x + 2} = 0 \). Here it is: \( \frac{x^2 + x - 2}{x + 2} = 0 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x + 2)(x - 1) = 0 \Rightarrow x = -2 \) or \( x = 1 \). Therefore \( x = -2 \) and \( x = 1 \) are both solutions. Is my proof correct?

The argument that \( \left( \frac{x^2 + x - 2}{x + 2} = 0 \right) \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x + 2)(x - 1) = 0 \Rightarrow (x = -2 \) or \( x = 1 \) \) is correct. However, the assertion that ”therefore \( x = -2 \) and \( x = 1 \) are both solutions” is not correct. The reason is this: The assertion that ”\( x = -2 \) and \( x = 1 \) are both solutions” is the converse of the assertion that \( \left( \frac{x^2 + x - 2}{x + 2} = 0 \right) \Rightarrow (x = -2 \) or \( x = 1 \) \). A statement can be true while its converse is false, and that is the case here.

**IF AND ONLY IF**

”\( p \) if and only if \( q \)” is called a biconditional. It expresses the idea that the two statements \( p \) and \( q \) have the same truth value. Therefore the statement is true when, and only when, \( p \) and \( q \) are both true or both false.

Thus the statement “\( 2 + 2 = 4 \) if and only if \( 9 = 3 \times 3 \)” is true. Never mind that you see no rhyme or reason for putting the two parts together. When two sentences are combined by using the words “if and only if”, the result is true when the two are both true or when they are both false.

You may use the notation \( p \iff q \) to stand for “\( p \) if and only if \( q \)”;

but you must use it correctly.

What about the following sentence? Is it true or false?

\( 7 < 5 \iff \) Garth Brooks is Harry Potter.

Both component parts are false, therefore the entire statement is true.

The sentence \( (2 + 2 = 4) \iff (1 = 0) \) is false because the parts do not have the same truth value.

**Logically equivalent statements.**

Two statements are called logically equivalent if one is true when, and only when, the other is true. It should be clear to you that \( p \) and \( q \) are logically equivalent exactly when \( p \iff q \) is true. Here are some important examples:

(I) \( (a) \ p \Rightarrow q \) and \( (b) \ q \lor \sim p \) are logically equivalent. (Hint: They are both false in exactly one way.)
Exercise 11 Write statements (1) and (2) above (in exercise 8) in the form of (b) above without the symbols ⇒ or ∼.

1 ≤ 2 OR 2 + 2 ≠ 5; 3 is an even number OR 2 + 2 ≠ 5

(II) (a) ∼ (p ⇒ q) and (b) p AND ∼ q are logically equivalent. (Hint: They are both true in exactly one way.)

Exercise 12 Write the negation of statements (1) and (2) (in exercise 8) in the form of (b) above without the symbols ⇒ or ∼.

1 > 2 AND 2 + 2 = 5; 3 is an odd number AND 2 + 2 = 5

(III) The statement ∼ q ⇒ ∼ p is called the contrapositive of p ⇒ q. They are logically equivalent. (Hint: They are both false in exactly one way.) It is often a very useful strategy to prove the contrapositive of a statement rather than the statement itself.

Exercise 13 Suppose that n is a natural number. Prove that if n² is even then n is even. It is not clear at all how one would prove this directly, but proving the contrapositive is not hard. State the contrapositive and prove it.

Proof was given in class.

Exercise 14 Write the contrapositives of statements (1) and (2) (in exercise 8) without the symbols ⇒ or ∼.

If 1 > 2 then 2 + 2 ≠ 5; If 3 is an odd number then 2 + 2 ≠ 5

(IV) The statement q ⇒ p is called the converse of p ⇒ q. The two are NOT logically equivalent.

Exercise 15 Write the converses of statements (1) and (2) in exercise 8 without the symbols ⇒ or ∼.

If 1 ≤ 2 then 2 + 2 = 5; If 3 is an even number then 2 + 2 = 5.

Proof by Contradiction

Suppose I want to prove that a statement p is true. A proof by contradiction goes like this: Assume p is false and deduce a contradiction – that is, deduce something that is known to be false.

Here’s what this means in logical symbols. Suppose r is a statement that I know is false. I prove (∼ p) ⇒ r is true. Since r is a statement that I know is false I claim that this proves (∼ p) must be false and therefore p must be true!
Example: Suppose I want to prove that the statement \( p \Rightarrow q \) is true by contradiction. I assume it’s false and try to arrive at a contradiction. But the negation of \( p \Rightarrow q \) is logically equivalent to \((p \text{ AND } \sim q)\). This means we are assuming \( p \) is true and \( q \) is false and from this we want to derive a third statement \( r \) that is false (this is the contradiction).

Summary: To prove that the statement \( p \Rightarrow q \) is true by contradiction, assume \( p \) is true and \( q \) is false and deduce a contradiction – that is, deduce something that is known to be false.

Exercise 16 Proving \( p \Rightarrow q \) by contradiction seems very much like proving the contrapositive of \( p \Rightarrow q \). Explain the difference.

Proving the contrapositive means assuming \( \sim q \) and proving \( \sim p \). A proof by contradiction means assuming both \( \sim q \) and \( p \) and deriving a statement that is false.

Logical Quantifiers.

\( \forall \) stands for “for every” or “for all.” \( \exists \) stands for “there exists.”

We often use the phrase “such that” with “there exists.” In that case we will use \( \ni \) as shorthand for “such that.” For example, the sentence, “There exists a positive number” may be written as “There exists a number \( x \) such that \( x > 0 \)” which may also be written as “\( \exists x \ni x > 0 \).”

You may use the symbols \( \forall, \exists \) and \( \ni \), but you must use them correctly. They are often misused, so if you are not sure just write out the words.

Examples (Unless otherwise specified, assume \( x \) and \( y \) are real numbers.)

(a) The square of any real number is non-negative: \( \forall x, x^2 \geq 0 \). If it were not clear that \( x \) is a real number you could write: \( \forall \text{ real number } x, x^2 \geq 0 \).

(b) Every non-negative real number has a square root: \( \forall x \geq 0, \exists y \ni y^2 = x \). Here’s another way: \( \forall x, (x \geq 0 \Rightarrow \exists y \ni y^2 = x) \).

(c) If \( x + 1 = y + 1 \), then \( x = y \): \( \forall x \forall y, (x + 1 = y + 1 \Rightarrow x = y) \). Here’s another way: \( \forall x, y, (x + 1 = y + 1 \Rightarrow x = y) \).

Exercise 17 Express the negation of the statements (a), (b) and (c) above in words and using \( \forall \) and \( \exists \). (It is not acceptable to just write “not” or “it is not the case that” in front of the statement.) Which of the six statements are true?

(a) \( \exists x \ni x^2 < 0 \). (b) \( \exists x \geq 0 \ni \forall y, y^2 \neq x \). (c) \( \exists x, y \ni x + 1 = y + 1 \text{ AND } x \neq y \).

Exercise 18 Express the following statements using \( \forall \) and \( \exists \), then express the negation of these statements (1) in words and (2) using \( \forall \) and \( \exists \). (It is not acceptable to just write “not” or “it
is not the case that” in front of the statement.) Which of the statements are true? Unless otherwise stipulated, assume $x$, $a$, and $b$ are real numbers in all the following.

(i) Every real number has a cube root. (True)

$\forall x \exists y \ni y^3 = x$. Negation: $\exists x \forall y, y^3 \neq x$.

(ii) There is a number $x$ whose reciprocal is zero. (False)

$\exists x \ni \frac{1}{x} = 0$. Negation: $\forall x, \frac{1}{x} \neq 0$.

(iii) $7x = 14$ has an integer solution. (True)

$\exists n \in \mathbb{N} \ni 7n = 14$. Negation: $\forall n \in \mathbb{N}, 7n \neq 14$.

(iv) For every number $a$, there exists a number $b$ such that $ax^2 = b$ has a solution. (False, consider $a = 0$)

$\forall a \exists b \exists x \ni ax^2 = b$. Negation: $\exists a \exists b, x, ax^2 \neq b$.

(v) There exists a number $b$ such that for every number $a$, $ax^2 = b$ has a solution. (True, consider $b = 0$)

$\exists b \forall a, \exists x \ni ax^2 = b$. Negation: $\forall b \exists a \forall x, ax^2 \neq b$.

(vi) For any two numbers $a$ and $b$, $x + a = b$ has a solution. (True)

$\forall a, b, \exists x \ni x + a = b$. Negation: $\exists a, b \forall x, x + a \neq b$. 

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