Solutions

1. \[ \lim_{x \to a} (f(x) - f(a)) = \lim_{x \to a} \left[ \frac{f(x) - f(a)}{x - a} \right] \]
   \[= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) \]
   \[= f'(a) \cdot 0 = 0. \text{ Therefore } \lim_{x \to a} f(x) = f(a). \quad \text{Q.E.D.} \]

2. \[ f'(a) = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} = \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})} = \]
   \[\lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} = \frac{1}{2\sqrt{a}}. \quad \text{Q.E.D.} \]

3. The case \( n = 1 \) says that \( f \) is differentiable at \( a \) and \( f'(a) = f(a) + f'(a) \), which is true. Assume it's true for some particular \( n \). Show that
   \[(D^{n+1})f(a) = (n+1)f'(a). \]

   \[D^{n+1}f(a) = D[f^n(a)](a) = \]
   \[(Df^n)(a) + (Df^n)(a)f^n(a) \text{ by the product rule. This is} \]
   \[f'(a)f^n(a) + nf(a)f^{n-1}(a) \text{ by the induction hypothesis. This is} \]
   \[f'(a)[f^n(a) + nf(a)] = \]
   \[f'(a)(n+1)f^n(a). \quad \text{Q.E.D.} \]
(4) (A) \( \forall M \in \mathbb{R} \ \exists \delta > 0 \ \exists x > r \Rightarrow g(x) \leq M. \)
(B) \( \forall M \in \mathbb{R} \ \exists N \in \mathbb{R} \ \exists x < N \Rightarrow A(x) \geq M. \)
(C) \( \forall \varepsilon > 0 \ \exists \delta > 0 \ \exists y < b \Rightarrow |f(x) - y| < \varepsilon. \)

(5) Suppose to the contrary that \( a < M. \)
Let \( \varepsilon = \frac{M-a}{2}. \) Then \( x_n \to a \) implies that:
\[ \exists N \in \mathbb{N} \ \forall n \geq N \Rightarrow |x_n - a| < \varepsilon. \]
Thus, \( a - \varepsilon < x_n < a + \varepsilon = a + \frac{M-a}{2} = \frac{M+a}{2} \leq M. \)
But \( x_n < M \) do a contradiction. \( \square \)

(6) Suppose \( L > M. \) Let \( \varepsilon = \frac{L-M}{2}. \) Since:
\[ \lim_{x \to a} f(x) = L, \ \exists \delta > 0 \ \exists \delta < |x - a| < \varepsilon \text{ and } x \in I \Rightarrow \]
\[ |f(x) - L| < \varepsilon, \text{ which says } L - \varepsilon < f(x) < L + \varepsilon. \]
But \( L - \varepsilon = L - \frac{L-M}{2} = \frac{L+M}{2} > L. \)
Thus, we see that \( \forall \delta > 0 \ \exists \delta < |x - a| < L \text{ and } x \in I \Rightarrow \]
\[ f(x) > L - \varepsilon > L, \text{ a contradiction.} \]

(7) Let \( \varepsilon > 0 \) be given. Since \( \lim_{x \to a} g(x) = 0, \) \( \exists \delta > 0 \ \exists \delta < |x - a| < \varepsilon \Rightarrow \]
\[ |g(x)| < \frac{\varepsilon}{M}. \text{ If } 0 < |x - a| < \delta, \text{ then } \]
\[ |f(x)g(x)| \leq M|g(x)| < M \left( \frac{\varepsilon}{M} \right) = \varepsilon. \ \square \]
8. \( \forall \varepsilon > 0 \exists x \in E \ni \inf E - \varepsilon < x \leq \inf E. \)
Let \( \varepsilon = \frac{1}{n} \), then \( \exists x_n \in E \ni \inf E - \frac{1}{n} < x_n < \inf E. \)
Since \( \lim_{n \to \infty} (\inf E - \frac{1}{n}) = \inf E \), we see that \( \lim x_n = \inf E \) by the squeeze theorem.

9. Let \( \{x_n\} \) converge to \( L \). Let \( \varepsilon > 0 \) be given.
\( \exists N \in \mathbb{N} \ni |x_n - L| < \frac{\varepsilon}{2} \forall n \geq N. \)
Let \( n, m \geq N. \) Then \( |x_n - x_m| = |x_n - L + L - x_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \), Q.E.D.

10. Let \( \varepsilon > 0 \) be given. Since \( \{x_n\} \) is Cauchy,
\( \exists N_1 \in \mathbb{N} \ni |x_n - x_m| < \frac{\varepsilon}{2} \) for \( n, m \geq N_1. \)
Likewise \( \exists N_2 \in \mathbb{N} \ni |y_n - y_m| < \frac{\varepsilon}{2} \) for \( n, m \geq N_2. \)
Let \( N = \max(N_1, N_2) \) and let \( n, m \geq N. \)
Then \( n, m \geq N_1 \) and \( N_2 \) so both \( |x_n - x_m| < \frac{\varepsilon}{2} \) and \( |y_n - y_m| < \frac{\varepsilon}{2} \); therefore
\[ |(x_n + y_n) - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]
Q.E.D.
11. Let $\varepsilon > 0$ be given. Fix $x_0$ in Cauchy so

$$\exists N \geq n \geq N \Rightarrow |x_n - x_m| < \frac{\varepsilon}{|c|}.$$ 

$$\Rightarrow |c||x_n - x_m| < \varepsilon$$

$$\Rightarrow |cx_n - cx_m| < \varepsilon. \quad \text{Q.E.D.}$$

12. Let $\varepsilon > 0$ be given. Since $\{y_n\}$ is bounded,

$$\exists M \geq y_n \leq M \forall n \in \mathbb{N},$$

Since $x_n \to 0$, \exists

$$N \geq n \geq N \Rightarrow |x_n| \leq \frac{\varepsilon}{M}.$$ 

Let $n \geq N$, then $|x_n y_n| \leq |x_n| |M| \leq \frac{\varepsilon}{M} \cdot M = \varepsilon.$

Q.E.D.

13. Let $M \in \mathbb{R}$. Since $\{y_n\}$ is bounded \exists

$$K \in \mathbb{R} \ni |y_n| \leq K \forall n \in \mathbb{N},$$

Since $x_n \to x \Rightarrow \exists$ \exists

$$N \geq n \geq N \Rightarrow |x_n| \leq M + K.$$ 

Let $n \geq N,$

Then $|x_n + y_n| = |x_n| - 1y_n| \leq M + K - K = M.$

Q.E.D.

14. Let $\varepsilon > 0$ be given. \exists $\delta_1 > 0$ and $\delta_2 > 0 \Rightarrow$

$$|f(x) - f(a)| < \frac{\varepsilon}{2} \text{ if } |x - a| < \delta_1 \text{ and } |g(x) - g(a)| < \frac{\varepsilon}{2} \text{ if } |x - a| < \delta_2.$$ 

Let $\delta = \min \{\delta_1, \delta_2\}$. If

$|x - a| < \delta$ then $|f(x) - f(a)| < \frac{\varepsilon}{2}$ and

$|g(x) - g(a)| < \frac{\varepsilon}{2}$. By the triangle inequality

$$|f(x) + g(x) - (f(a) + g(a))| \leq |f(x) - f(a)| + |g(x) - g(a)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \text{Q.E.D.}$$

Now assume $c \neq 0$, \exists $\delta > 0$

$\exists |f(x) - f(a)| < \frac{\varepsilon}{|c|}$ if $|x - a| < \delta$. Therefore

$|cf(x) - cf(a)| = |c||f(x) - f(a)| < |c|\frac{\varepsilon}{|c|} = \varepsilon.$

If $|x - a| < \delta$. The case $c = 0$ is trivial. \quad \text{Q.E.D.}
Let \( \{a_n\} \) be a sequence converging to \( a \).
Then \( f(a_n) \to f(a) \) because \( f \) is continuous at \( a \). But \( g \) is continuous on \( B \), so \( g \) is continuous at \( f(a) \in B \). Therefore
\[
g(f(a_n)) \to g(f(a))
\]
Thus,
\[
g(f(a_n)) \to g(f(a)).
\]
Since \( g(f(a_n)) \to g(f(a)) \) if sequence \( \{a_n\} \) converging to \( a \),
we conclude that
\[
l_{n}g(f(a)) = g(f(a)), \text{ so } g \text{ is continuous at } a.
\]
14. Let $f$ and $g$ be uniformly continuous on an interval $I$ and let $c \in \mathbb{R}$. Prove that $f + g$ and $cf$ are uniformly continuous on $I$. Use only the definition.

15. Let $A$ and $B$ be open intervals. Let $f : A \to B$ and $g : B \to C$. Suppose that $f$ is continuous at $a \in A$ and $g$ is continuous at $f(a) \in B$. Use sequences to prove that $g \circ f$ is continuous at $a \in A$.

16. Fill in the blanks below for the proof. Let $A$ and $B$ be open intervals. Let $f : A \to B$ and $g : B \to C$. Suppose that $f$ is continuous at $a \in A$ and $g$ is continuous at $b = f(a) \in B$. Use the definitions only (no sequences) to prove that $g \circ f$ is continuous at $a \in A$.

**Proof:** Let $\epsilon > 0$ be given. We want to find $\delta > 0$ so that if $|x - a| < \delta$ then $|g(f(x)) - g(f(a))| < \epsilon$.

Since $g$ is continuous at $b \in B$ there's a $\delta_1 > 0$ so that if $y \in B$ and $|y - b| < \delta_1$ then $|g(y) - g(b)| < \frac{\epsilon}{|f'|}$.

But $f$ is continuous at $a$ so there's a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \delta_1$.

This is the $\delta$ we are looking for. Now let's show it works. Let $|x - a| < \delta$.

Then $|f(x) - f(a)| < \frac{\epsilon}{|f'|}$. Now $f(a) = b$ so, letting $f(x) = y$, this says $|y - b| < \frac{\epsilon}{|f'|}$.

But this implies that $|g(y) - g(b)| < \frac{\epsilon}{|f'|}$. Substitute $f(x)$ for $y$ and $f(a)$ for $b$ in $|g(y) - g(b)| < \epsilon$ to get $|g(f(x)) - g(f(a))| < \frac{\epsilon}{|f'|}$ or $|g \circ f(x) - g \circ f(a)| < \epsilon$.

17. Fill in the blanks below for the proof. If $f$ is continuous at $x_0$ and $f(x_0) > 0$ then there exists an open interval $(a, b)$ containing $x_0$ and a number $\epsilon > 0$ such that $f(x) \geq \epsilon$ for every $x \in (a, b)$.

**PROOF:** Let $\epsilon = f(x_0)/2$. Since the function $f$ is continuous at $x_0$ there exists a number $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \frac{\epsilon}{|f'|}$.

For the open interval $(a, b)$ let $a = x_0 - \frac{\delta}{|f'|}$ and $b = x_0 + \frac{\delta}{|f'|}$. If $x \in (a, b)$ then $|x - x_0| < \delta$, so $|f(x) - f(x_0)| < \epsilon = f(x_0)/2$. Therefore if $x \in (a, b)$ then $f(x) = f(x_0)/2$. But $f(x_0) - \epsilon = \frac{\epsilon}{|f'|}$ so $f(x) > \frac{\epsilon}{|f'|}$ QED.

18. Fill in the blanks below for the proof. The Extreme Value Theorem Let $f$ be continuous on the closed interval $I = [a, b]$. Then (a) $f$ is bounded on $I$ and (b) there exists a point $c$ in the interval $I$ such that $f(c) = \inf \{f(x) | x \in I\}$.

**Proof of (a):**

Suppose the function is not bounded on the interval $I$. Then for each positive integer $n$ there exists an element $x_n$ in $I$ such that $f(x_n) > \frac{n}{k}$.

But since the sequence $x_n$ is contained in the interval $I$, it is bounded. Therefore, by the Bolzano-Weierstrass theorem it has a subsequence $x_{n_k}$ that converges to some point $p$ in $I$. However, $f$ is continuous at the point $p$, so $f(x_{n_k}) \to f(p)$.

Since $f(x_{n_k})$ is convergent $\sum_{k=1}^{\infty} \frac{f(x_{n_k})}{k}$ is bounded. But $f(x_{n_k}) > \frac{n}{k}$ for each $k \in \mathbb{N}$. This is a contradiction.
Proof of (b):

Let $S = \{ f(x) | x \in I \}$. $S$ is bounded by part (a) therefore it has a infimum, call it $m$.

By problem number two in section 2.3 (which is #8 above) there is a sequence $y_n$ in $S$ that converges to $m$.

But since $y_n$ belongs to $S$, there must exist a number $x_n$ in $I$ such that $y_n = f(x_n)$.

Now $\{x_n\}$ is a sequence in the closed bounded interval $I$. Therefore by the Bolzano-Weierstrass theorem there must be a subsequence $\{x_{n_k}\}$ converging to some number $c$ (c must be in $I$ by problem #5 above).

But $f$ is continuous on $I$ therefore $f(x_{n_k}) \to f(c)$.

However, $f(x_{n_k}) = y_{n_k}$ by definition and $y_n \to m$. Therefore, the subsequence $f(x_{n_k}) = y_{n_k}$ also converges to $m$.

In conclusion, we have observed that $f(x_{n_k}) \to f(c)$ and $f(x_{n_k}) \to m$. Therefore, $m = f(c)$.

19. Fill in the blanks below for the proof. **The Intermediate Value Theorem:** Suppose $a < b$ and $f(a) < 0$ and $f(b) > 0$. Then there is a number $c$ between $a$ and $b$ such that $f(c) = 0$.

**PROOF:** Let $S = \{ x \in [a, b] | f(x) > 0 \}$

$S$ is not empty because we are given that $f(b) > 0$ and therefore $b$ is in $S$.

$S$ is bounded below because every element $x$ in $S$ is greater than or equal to $a$.

Since $S$ is bounded below, it has an infimum $c$ by the completeness axiom. We will now show that $f(c) = 0$.

By Problem #8 above there is a sequence $\{x_n\}$ in $S$ converging to $c$.

Since the function $f$ is continuous at $c$, $f(x_n) \to f(c)$.

Since $x_n$ belongs to $S$, $f(x_n) > 0$ for every $n$.

Therefore by problem #5 above, $f(c) \geq 0$.

Now we will show that $f(c)$ cannot be positive. This will prove that $f(c) = 0$. We will do this by contradiction. Thus we suppose that $f(c) > 0$. By problem #16 above, there’s an open interval $(u, v)$ containing $c$ such that $f(x) > 0$ for every $x$ in $(u, v)$.

Let $d = \frac{u + c}{2}$. Then $u < d < c < v$.

Therefore we can conclude that $f(d) > 0$ because $d$ is in the interval $(u, v)$. Therefore $d \in S$ because $f(d) > 0$.

Thus we have found a number $d$ in $S$ such that $d < c$. But this contradicts the fact that $c$ is \( \inf S \). QED.