NOTES FOR DAYS 6 & 7

Proof by Mathematical Induction. Chapter 5. DAY 8 Exam 1

Let \( P(n) \) be a proposition about the natural number \( n \). Mathematical Induction says the following:

If \( P(1) \) is true and if \( [P(n) \Rightarrow P(n+1)] \) is true for all natural numbers \( n \) then \( P(n) \) is true for every natural number \( n \).

Be aware that mathematical induction is actually an axiom of the natural numbers.

The following is called the **Principle of Strong Induction.** (Sometimes ordinary induction is referred to as “weak induction”.)

If \( P(1) \) is true if \( [(P(1) \ AND \ ... \ AND \ P(n)) \Rightarrow P(n+1)] \) is true for all natural numbers \( n \) then \( P(n) \) is true for every natural number \( n \).

Strong induction follows from mathematical induction. **Here’s the proof.**

Let \( Q(n) \) be the statement \( P(1) \ AND \ ... \ AND \ P(n) \).

Then we want to show that if \( P(1) \) is true and \( [Q(n) \Rightarrow P(n+1)] \) is true for all natural numbers \( n \) then \( P(n) \) is true for every natural number \( n \).

What we’re going to do is use mathematical induction to show that \( Q(n) \) is true for every natural number \( n \). Since \( Q(n) \Rightarrow P(n) \) it will follow that \( P(n) \) is true for every natural number \( n \).

First, notice that \( Q(1) \) is the same as \( P(1) \). Therefore, since \( P(1) \) is assumed to be true, \( Q(1) \) is true.

Also for each natural number \( n \), \( [Q(n) \Rightarrow P(n+1)] \). Thus for each natural number \( n \), \( [Q(n) \Rightarrow (Q(n) \ AND \ P(n+1)) \iff Q(n+1)] \).

Since \( Q(1) \) is true and \( [Q(n) \Rightarrow Q(n+1)] \) is true for each natural number \( n \), it follows that \( Q(n) \) is true for all \( n \) by induction. QED.

**Exercise 19** Prove these interesting facts using mathematical induction.

1. For all natural numbers \( n \), \( \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \)

2. For all natural numbers \( n \), \( \sum_{j=1}^{n} j^3 = \left( \frac{n(n+1)}{2} \right)^2 \)
3. Suppose $a \neq 1$ and $a \neq 0$. For all natural numbers $n$, \[
\sum_{i=1}^{n} a^{i-1} = \frac{1 - a^n}{1 - a}\]

4. If $a$ is a real number with $a > -1$, then for all natural numbers $k$, $(1 + a)^k \geq 1 + ak$

5. For all natural numbers $k$, $9^k - 8k - 1$ is divisible by 64.

6. $5^n - 4n - 1$ is divisible by 8 for every natural number $n$.

7. Show that if $2 + 4 + 6 + \ldots + 2n = n(n + 1) + 2$ is true for $n = n_0$, then it is true for $n = n_0 + 1$. Is the formula true for all $n$?

8. The set of natural numbers (denoted $\mathbb{N}$) is what is called a “well-ordered set”. What this means is that every nonempty subset of $\mathbb{N}$ has any smallest element. Use the principle of strong induction to prove this.

   Hint: Suppose $S$ is a nonempty set of natural numbers with no least element, and let $T$ be the set of numbers in $\mathbb{N}$ that are not in $S$. Use strong induction to prove that $T = \mathbb{N}$. This contradicts the fact that the set $S$ is nonempty.

9. Try to prove the above theorem using only “weak” induction, and not strong induction.

22. (a) If $c > 1$, show that $c^n > c$ for all $n \in \mathbb{N}$, and that $c^n > c$ for $n > 1$.
   (b) If $0 < c < 1$, show that $c^n \leq c$ for all $n \in \mathbb{N}$, and that $c^n < c$ for $n > 1$.

23. If $a > 0$, $b > 0$ and $n \in \mathbb{N}$, show that $a < b$ if and only if $a^n < b^n$. [Hint: Use Mathematical Induction].

24. (a) If $c > 1$ and $m, n \in \mathbb{N}$, show that $c^m > c^n$ if and only if $m > n$.
   (b) If $0 < c < 1$ and $m, n \in \mathbb{N}$, show that $c^m < c^n$ if and only if $m > n$.

25. Assuming the existence of roots, show that if $c > 1$, then $c^{1/m} < c^{1/n}$ if and only if $m > n$.

26. Use Mathematical Induction to show that if $a \in \mathbb{R}$ and $m, n \in \mathbb{N}$, then $a^{m+n} = a^m a^n$ an $(a^m)^n = a^{mn}$.