Sets. Chapter 6

In supplementary exercises 5.25 and 5.26, one sees \( \bigcup_{B \in \mathcal{B}} B \) and \( \bigcap_{B \in \mathcal{B}} B \). These may also be written as \( \bigcup_{B \in \mathcal{B}} B \) and \( \bigcap_{B \in \mathcal{B}} B \) or simply as \( \bigcup B \) and \( \bigcap B \).

If the sets in \( \mathcal{B} \) are indexed as, for example, \( B_i, i \in I \) then one may see \( \bigcup_{i \in I} B_i \) and \( \bigcap_{i \in I} B_i \). If \( I = \mathbb{N} \) one may see \( \bigcup_{i=1}^{\infty} B_i \) and \( \bigcap_{i=1}^{\infty} B_i \).

**Exercise 20** In supplementary exercise 5.25 (a), (b) and (c), write \( \bigcup_{B \in \mathcal{B}} B \) and \( \bigcap_{B \in \mathcal{B}} B \) in the other forms defined above.

**Exercise 21** Let \( S \) be the set of all sets that do not contain themselves as elements. \( (A \in S \iff A \not\in A) \). Is \( S \in S \)? Is \( S \not\in S \)? Here's a popular version of this: In a town is a barber who shaves every man who does not shave himself. Who shaves the barber?

**THEOREM.**
Let \( \{A_i \mid i \in I\} \) be an indexed family of sets, and let \( B \) be any set, all subsets of some universal set \( U \). Then:

1. \( B \cup \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cup A_i) \)
2. \( B \cap \bigcap_{i \in I} A_i = \bigcap_{i \in I} (B \cap A_i) \)
3. \( B \cap \bigcup_{i \in I} A_i = \bigcup_{i \in I} (B \cap A_i) \)
4. \( B \cup \bigcap_{i \in I} A_i = \bigcap_{i \in I} (B \cup A_i) \)
5. \( \left( \bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c \)
6. \( \left( \bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c \)

[\( \varnothing \)] Let \( \{A_i \mid i \in I\} \) be an indexed family of sets. Then for any \( i_0 \in I \)

\( A_{i_0} \subseteq \bigcup_{i \in I} A_i \) and \( \bigcap_{i \in I} A_i \subseteq A_{i_0} \).
5.8 Let \( S = \{ \emptyset, \{ \emptyset \} \} \). Determine whether each of the following is True or False. Explain your answers.
(a) \( \emptyset \subseteq S \)  
(b) \( \emptyset \in S \)  
(c) \( \{ \emptyset \} \subseteq S \)  
(d) \( \{ \emptyset \} \in S \)

5.9 Fill in the blanks in the following proof.

**THEOREM:** Let \( A \) be a subset of \( U \). Then \( A \cup (U \setminus A) = U \).

**Proof:** If \( x \in A \cup (U \setminus A) \), then \( x \in \) \( \) or \( x \in \) \( \). Since both \( A \) and \( U \setminus A \) are subsets of \( U \), in either case we have \( \). Thus \( \subseteq \) \( \). On the other hand, suppose that \( x \in \). Now either \( x \in A \) or \( x \notin A \). If \( x \notin A \), then \( x \in \). In either case \( x \in \). Hence \( \subseteq \). \( \star \)

5.10 Fill in the blanks in the proof of the following theorem.

**THEOREM:** \( A \subseteq B \) iff \( A \cup B = B \).

**Proof:** Suppose that \( A \subseteq B \). If \( x \in A \cup B \), then \( x \in \) or \( x \in \). Since \( A \subseteq B \), in either case we have \( x \in B \). Thus \( \subseteq \). On the other hand, if \( x \in \), then \( x \in A \cup B \), so \( \subseteq \). Hence \( A \cup B = B \).

Conversely, suppose that \( A \cup B = B \). If \( x \in A \), then \( x \in \). But \( A \cup B = B \), so \( x \in \). Thus \( \subseteq \). \( \star \)

5.11 Fill in the blanks in the proof of the following theorem.

**THEOREM:** \( A \subseteq B \) iff \( A \cap B = A \).

**Proof:** Suppose that \( A \subseteq B \). If \( x \in A \cap B \), then clearly \( x \in A \). Thus \( A \cap B \subseteq A \). On the other hand, \( \subseteq \) \( \). Thus \( A \subseteq A \cap B \), and we conclude that \( A \cap B = A \).

Conversely, suppose that \( A \cap B = A \). If \( x \in A \), then \( x \in \).

Thus \( A \subseteq B \). \( \star \)

5.12 Suppose you are to prove that set \( A \) is a subset of set \( B \). Write a reasonable beginning sentence for the proof, and indicate what you would have to show in order to finish the proof.

5.15 Which statement(s) below would enable one to conclude that \( x \in A \cap B \)?
(a) \( x \in A \) and \( x \in B \).
(b) \( x \in A \) or \( x \in B \).
(c) \( x \in A \) and \( x \notin A \setminus B \).
(d) \( x \notin A \), then \( x \in B \).

5.16 Which statement(s) below would enable one to conclude that \( x \notin A \setminus B \)?
(a) \( x \in A \) and \( x \notin B \).
(b) \( x \in A \) or \( x \notin B \).
(c) \( x \in A \) and \( x \notin A \setminus B \).
(d) \( x \in A \) and \( x \notin A \cap B \).

5.17 Which statement(s) below would enable one to conclude that \( x \notin A \setminus B \)?
(a) \( x \notin A \setminus B \).
(b) \( x \notin B \setminus A \).
(c) \( x \in A \cap B \).
(d) \( x \in A \setminus B \) and \( x \in A \).
(e) \( x \in A \setminus B \) and \( x \in A \cap B \).

5.19 Prove: If \( U = A \cup B \) and \( A \cap B = \emptyset \), then \( A = U \setminus B \).

5.20 Prove: \( A \cap B \) and \( A \setminus B \) are disjoint and \( A = (A \cap B) \cup (A \setminus B) \).

5.21 Prove or give a counterexample: \( A \setminus (A \setminus B) = B \setminus (B \setminus A) \).

5.22 Prove or give a counterexample: \( A \setminus (B \setminus A) = B \setminus (A \setminus B) \).
5.23 Let \( A \) and \( B \) be subsets of a universal set \( U \). Prove the following.

(a) \( A \setminus B = (U \setminus B) \setminus (U \setminus A) \)
(b) \( U \setminus (A \setminus B) = (U \setminus A) \cup B \)
(c) \( (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B) \)

5.25 Find \( \bigcup_{B \in \mathcal{B}} B \) and \( \bigcap_{B \in \mathcal{B}} B \) for each collection \( \mathcal{B} \).

(a) \( \mathcal{B} = \left\{ \left(1, 1 + \frac{1}{n}\right) : n \in \mathbb{N} \right\} \)
(b) \( \mathcal{B} = \left\{ \left(1, 1 + \frac{1}{n}\right) : n \in \mathbb{N} \right\} \)
(c) \( \mathcal{B} = \{ [2, x) : x \in \mathbb{R} \text{ and } x > 2 \} \)
(d) \( \mathcal{B} = \{ [0, 3), (1, 5), [2, 4) \} \)

5.26 Let \( \{ A_j : j \in J \} \) be an indexed family of sets and let \( B \) be a set. Prove the following generalizations of Theorem 5.13.

(a) \( B \cup \bigcap_{j \in J} A_j = \bigcap_{j \in J} (B \cup A_j) \)
(b) \( B \cap \bigcup_{j \in J} A_j = \bigcup_{j \in J} (B \cap A_j) \)
(c) \( B \setminus \bigcap_{j \in J} A_j = \bigcap_{j \in J} (B \setminus A_j) \)
(d) \( B \setminus \bigcup_{j \in J} A_j = \bigcup_{j \in J} (B \setminus A_j) \)