1. Find the matrix products: 
\[
\begin{bmatrix}
2 & 3 & 1 \\
1 & -1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 3 \\
-1 & 2
\end{bmatrix}
\text{ and }
\begin{bmatrix}
1 & 3 \\
-1 & 2
\end{bmatrix}
\begin{bmatrix}
2 & 3 & 1 \\
1 & -1 & 0
\end{bmatrix}.
\]

2. What 2 by 2 matrix rotates a vector clockwise through a 90 degree angle? (See Section 2.1, #17). How about counterclockwise through a 90 degree angle? How about counterclockwise through a 45 degree angle?

3. Find the transpose of 
\[
\begin{bmatrix}
2 & 3 & 1 \\
1 & -1 & 0
\end{bmatrix}
\text{ and of }
\begin{bmatrix}
2 \\
1
\end{bmatrix}.
\]

4. Express the linear combination 
\[x_1(7, 2) + x_2(9, 4) + x_3(8, 6)\] in the form of matrix multiplication. (Respect rows and columns.)

5. Express the linear combination 
\[a \begin{bmatrix}
1 \\
2
\end{bmatrix} + b \begin{bmatrix}
4 \\
5
\end{bmatrix}\] in the form of matrix multiplication. (Respect rows and columns.)

6. Heathcliff claimed that the inverse of the matrix
\[
\begin{bmatrix}
1 & 4 & 0 \\
2 & 1 & -2 \\
-1 & 0 & 1
\end{bmatrix}
\]
is
\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & 1 & 2 \\
3 & 0 & -1
\end{bmatrix}.
\]
Mary said, “I can tell you’re wrong by just looking at this!” Explain how she knows so fast.

7. Find the inverse of
\[
\begin{bmatrix}
1 & 1 & -1 \\
5 & 2 & 2 \\
2 & 0 & 3
\end{bmatrix}
\]
and check your answer. (On the exam you must show your steps and check your answer.)

8. Let 
\[
M = \begin{bmatrix}
1 & 3 & 0 \\
2 & 1 & -2 \\
-1 & 0 & 1
\end{bmatrix}.
\]
Use the fact that 
\[
M^{-1} = \begin{bmatrix}
1 & -3 & -6 \\
0 & 1 & 2 \\
1 & -3 & -5
\end{bmatrix}
\]
to solve the system 
\[
\begin{align*}
x + 3y &= 11 \\
2x + y - 2z &= 0 \\
x - 3y - 6z &= 11
\end{align*}
\]
and the system 
\[
\begin{align*}
x &= \sqrt{2} \\
y + 2z &= 0 \\
x - 3y - 5z &= \sqrt{2}
\end{align*}
\]

9. Let 
\[
A = \begin{bmatrix}
-1 & 2 & -5 \\
3 & 4 & 5 \\
0 & 1 & 2
\end{bmatrix}.
\]
Show the steps in finding the LU factorization of A. What is the “U” in the LU factorization of A? What is the “L” in the LU factorization of A? Check your answer.

10. Let 
\[
A = \begin{bmatrix}
1 & 4 & 3 & 1 \\
-2 & -3 & -4 & 0 \\
2 & 5 & 5 & 2 \\
-1 & 0 & -1 & 2
\end{bmatrix}
\]
Show the steps in finding the LU factorization of A. What is the “U” in the LU factorization of A? What is the “L” in the LU factorization of A? Check your answer.

11. Suppose \(A = LU\), where 
\[
L = \begin{bmatrix}
1 & 0 & 0 \\
3 & 1 & 0 \\
2 & -2 & 1
\end{bmatrix}
\]and 
\[
U = \begin{bmatrix}
-1 & 2 & -5 \\
0 & 4 & 5 \\
0 & 0 & 2
\end{bmatrix}.
\]
Use this to solve \(Ax = \begin{bmatrix}
-12 \\
-4 \\
-80
\end{bmatrix}\). Check your final answer. (On the exam, you must show how the LU factorization is used by forward and back substitution, and you must check your final answer.)

12. Find a non-zero 2 by 2 matrix \(M\) such that \(M^2 = 0\).

13. Explain why a matrix with a column of zeros cannot have an inverse.
14. Assume $A$ and $B$ are invertible $n$ by $n$ matrices. Prove that $BA$ is invertible and that $(BA)^{-1} = A^{-1}B^{-1}$.

What is wrong with this “proof” of the problem above?

$$(BA)(BA)^{-1} = (BA)(A^{-1}B^{-1})$$
$$(BA)(BA)^{-1} = B(AA^{-1})B^{-1}$$
$$(BA)(BA)^{-1} = BB^{-1}$$
$$(BA)(BA)^{-1} = I$$
$I = I$
End of proof.

What is wrong with this “proof” of the problem above?

$$(BA)^{-1} = A^{-1}B^{-1}$$
$$(B^{-1}A^{-1} = (AB)^{-1})$$
$$(BB^{-1}A^{-1} = B(AB)^{-1})$$
$$(A^{-1} = B(AB)^{-1})$$
$I = I$
End of proof.

15. Assume $A$ and $B$ are $n$ by $n$ invertible matrices and that $AB$ is invertible. Prove that $A$ is invertible.

What is wrong with this “proof” of the problem above?

$$B^{-1}A^{-1} = (AB)^{-1}$$
$$(BB^{-1}A^{-1} = B(AB)^{-1})$$
$$(A^{-1} = B(AB)^{-1})$$
$I = I$
End of proof.

16. Assume $A$ is a square invertible matrix. Prove that $A^T$ is invertible and that $(A^T)^{-1} = (A^{-1})^T$.

17. Assume $A$, $B$ and $C$ are $n$ by $n$ matrices such that $BA = I$ and $AC = I$. Using only this fact, prove that $B = C$.

Do not use any matrix inverse notation like $A^{-1}$.

18. Find the angle between $(1, 1, 0)$ and $(2, 1, 3)$ in degrees.

19. Find a non-zero vector that is perpendicular to $(-1, 7, 2)$.

20. If $x \cdot y < 0$ what can you say about the angle between these vectors?

21. Suppose that $x$ is a vector in $\mathbb{R}^4$ of length 10 that makes an angle of 57 degrees with the unit vector $u$. If $\cos(57^\circ) = 0.5446$, find the dot product of $u$ and $x$.

22. If $x = (-1, 2)$ and $y = (3, 5)$ find $||x - y||$.

23. The graph of $y = 2x + 1$ in $\mathbb{R}^2$ is (a) a line, (b) a plane, (c) a point, or (d) none of these.

24. The graph of $y = 2x + 1$ in $\mathbb{R}^3$ is (a) a line, (b) a plane, (c) a point, or (d) none of these.

25. The graph of $xy = 1$ in $\mathbb{R}^3$ is (a) a line, (b) a plane, (c) a point, or (d) none of these.

26. The graph of $x^2 + y^2 = 0$ in $\mathbb{R}^3$ is (a) a line, (b) a plane, (c) a point, or (d) none of these.
27. On the given picture sketch:
   (a) all the points of the form \( ax + (1 - a)y \) with \( a \leq 0 \), and
   (b) all the points of the form \( ax + (1 - a)y \) with \( 0 \leq a \leq \frac{1}{2} \).

28. **TRUE - FALSE** In all the following assume \( A \) and \( B \) are 8 by 8 matrices.

1. __ The vectors \((-1, 2, 3)\) and \((5, 1, 1)\) are perpendicular.

2. __ \((A + B)(A - B) = A^2 - B^2\).

3. __ \((A - B)^2 = A^2 - 2AB + B^2\).

4. __ \((3, 1)^{-1} = (\frac{1}{3}, 1)\).

5. __ If \( A \) and \( B \) are invertible then so is \( BA^2 \).

6. __ If \( BA = I \) then \( A \) must equal \( B^{-1} \).

7. __ If \( A \) is invertible then \((A^{-1})^T = (A^T)^{-1}\).

8. __ If \( M \) is a 4 by 3 matrix then \( M^T \) is 3 by 4.

9. __ \((BA)^T = B^TA^T\).

10. __ The 4th row of \( AB \) is \( A \) times the 4th row of \( B \).

11. __ The 5th column of \( AB \) is the 5th column of \( A \) times \( B \).

12. __ If \( \bar{x} \) is an 8 by 1 vector then \( \bar{x}A \) is a linear combination of the rows of \( A \).

13. __ If \( \bar{x} \) is an 8 by 1 vector then \( A\bar{x} \) is a linear combination of the columns of \( A \).

14. __ A linear combination of two vectors always lies on the line joining them.

15. __ It is impossible for a system of linear equations to have exactly two solutions.

16. __ An elementary matrix is a matrix all of whose entries are ones and zeros.