1. Let \( S \) be a subspace of \( \mathbb{R}^n \). Show that the zero vector is the only vector in both \( S \) and \( S^\perp \). ANS: If \( x \) is in both \( S \) and \( S^\perp \) then \( x \cdot x = 0 \). This implies that \( x = 0 \).

2. Let \( A \) be any matrix. Show that if \( A^T A x = 0 \) then \( A x = 0 \). ANS: If \( A^T A x = 0 \) then \( A x \) is orthogonal to the rows of \( A^T \) and therefore to the row space of \( A^T \). Now the row space of \( A^T \) is the column space of \( A \) and so \( A x \) is orthogonal to the column space of \( A \). But \( A x \) is in the column space of \( A \) so it must be is orthogonal to itself. That is \( A x \cdot A x = 0 \). Therefore \( A x = 0 \).

3. Is the set of all 2 by 2 invertible matrices a subspace of all the 2 by 2 matrices? If not, explain why. If it is, what is its dimension? No. ANS: The zero matrix is not in the set.

4. Is the set of all 2 by 2 lower triangular matrices a subspace of all the 2 by 2 matrices? If not, explain why. If it is, what is its dimension? ANS: Yes. Dim = 3.

5. Is the set of all 2 by 2 singular matrices a subspace of all the 2 by 2 matrices? If not, explain why. If it is, what is its dimension? ANS: No.

\[
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}
\] and \[
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
\] are in the set but their sum is \( I \) which is not in the set.

6. Which sets below are subspaces of the vector space of all real-valued functions \( f \)? For those that are NOT subspaces, give an explicit example to show why.

(a) All functions \( f \) such that \( f(-1) = 0 \). YES (b) All functions \( f \) such that \( f(7) = 1 \). NO - the zero function is not in the set. (c) All functions \( f \) such that \( f(3) \geq 0 \). No. The constant function 1 is in the set but \(-1\) is not. (d) All functions \( f \) of the form \( f(x) = (ax)^2 + b \). NO (e) All functions \( f \) of the form \( f(x) = ax^2 + bx + a \). YES (f) All functions \( f \) such that \( f(4) = f(-5) \). YES.

7. What can be said about a system of linear equations \( A\bar{x} = \bar{0} \) if \( A \) has rank equal to the number of unknowns?

(a) It has exactly one solution. (b) It has no solutions. (c) Could be either (a) or (b). (d) It has infinitely many solutions. ANS: (a)

8. What can be said about a system of linear equations \( A\bar{x} = \bar{b} \) if \( A \) has rank equal to the number of unknowns and \( \bar{b} \neq \bar{0} \)?

(a) It has exactly one solution. (b) It has no solutions. (c) Could be either (a) or (b). (d) It has infinitely many solutions. ANS: (c)

9. If a homogeneous system of linear equations has more unknowns than equations what can be said about the number of solutions? (a) It has exactly one solution. (b) It has no solutions. (c) Either (a) or (b) may true. (d) It has infinitely many solutions. ANS: (d)

10. If a vector space \( V \) is spanned by some set of 9 vectors then \( V \) has dimension

(a) 9 or more (b) 9 or less (c) exactly 9 (d) could be either (a) or (b). ANS: (b)

11. If a vector space \( V \) contains 9 independent vectors then \( V \) has dimension

(a) 9 or more (b) 9 or less (c) exactly 9 (d) could be either (a) or (b). ANS: (a)

12. If \( M \) is a 9 by 6 matrix with rank 4, what are the dimensions of the four subspaces: row space of \( M \), null space of \( M \), column space of \( M \), left null space of \( M \)?

13. Put this matrix in reduced echelon form.

\[
\begin{bmatrix}
1 & -2 & -1 & 0 & 3 & 1 \\
0 & 0 & 0 & 1 & 4 & 1 \\
-1 & 2 & 2 & 0 & 1 & 0
\end{bmatrix}
\]

Based on the answer, what are the dimensions of the four associated subspaces?

\[
\begin{bmatrix}
1, -2, 0, 0, 7, 2 \\
0, 0, 1, 0, 4, 1 \\
0, 0, 0, 1, 4, 1
\end{bmatrix}
\]

\( \text{dim of col, row, null spaces} = 3, \text{dim of left null space} = 0. \)
14. Suppose \( S \) is the subspace of \( \mathbb{R}^5 \) spanned by \( v_1 = (2, -3, 1, 1, -1), v_2 = (1, 2, 0, 1, 2), \) and \( v_3 = (3, -8, 2, 1, -4). \) Find a basis for (a) \( S \) and (b) \( S^\perp. \)

(a) ANS: A basis for \( S \) is \( v_1 \) and \( v_2. \) Another answer is \((7, 0, 2, 5, 4) \) and \((0, 7, -1, 1, 5). \)
(b) ANS: \((-2, 1, 7, 0, 0), (-1, 0, 1, 1, 0), (-2, 0, 5, 0, 1)\).

15. Possible answers for the following are: (1) the row space of \( A, \) (2) the column space of \( A, \) (3) the null space of \( A, \) or (4) the left null space of \( A. \)

(a) The orthogonal complement of the row space of \( A \) is \( \text{null space of } A \).
(b) The orthogonal complement of the column space of \( A \) is \( \text{row space of } A \).
(c) The orthogonal complement of the null space of \( A \) is \( \text{column space of } A \).
(d) The orthogonal complement of the left null space of \( A \) is \( \text{column space of } A \).
(e) The orthogonal complement of the null space of \( A^T \) is \( \text{column space of } A \).

16. If \( A \) and \( B \) are matrices then the row space of \( AB \) is contained in the row space of \( A. \) The column space of \( AB \) is contained in the column space of \( A. \) The null space of \( AB \) contains the null space of \( A \).

The null space of \( AB \) contains the null space of \( A. \) ANS: \( B \) and \( b = 0. \)

17. What must be true about \( \tilde{b} \) in order for the set of solutions to \( A\tilde{x} = \tilde{b} \) to be a subspace? ANS: \( b = 0. \)

18. In order for \( A\tilde{x} = \tilde{b} \) to have a solution \( \tilde{b} \) must be in which one of the 4 subspaces associated with \( A? \) ANS: \( b \) is in the column space of \( A \)

19. Are the following sets of vectors independent or dependent? (You should be able to tell pretty fast by inspection without reducing any matrices.)

(a) \((\pi, 0, 0), (0, 0, \sqrt{7}), (0, 101, 0)\) INDEPENDENT. (b) \((\pi, 0, 0), (\sqrt{7}, 0, 0), (0, 101, 0)\) DEPENDENT (c) \((\pi, 12), (0, 0)\) DEPENDENT (d) \((\pi, \sqrt{3}, 1), (\sqrt{7}, 11/2, -172), (2, 91, \sqrt{91}), (13, 7, \pi)\) DEPENDENT (e) \(f(x) = 2x, g(x) = 7x^2, h(x) = 19\) INDEPENDENT. (f) \(f(x) = x + \sin^2 x, g(x) = -x + \cos^2 x, h(x) = 13\) DEPENDENT.

20. Assume that \( A = \begin{bmatrix} 1 & 1 & -1 & -3 & 6 & 0 & 0 \\ 1/2 & 3 & 9/2 & -43/2 & -8 & 0 & -1 \\ -5 & 4 & 23 & -57 & 1 & 0 & 67 \\ 7 & 1 & -19 & 27 & 2 & 0 & -64 \\ -1 & 0 & 3 & -5 & 1/3 & 0 & 34/3 \\ 1 & 5 & 7 & -35 & 2 & 0 & 12 \end{bmatrix} \)

reduces to \( R = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 & 0 & -10 \\ 0 & 1 & 2 & -8 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \)

(a) The rank of \( A \) is \( 3. \)
(b) The dimension of the column space of \( A \) is \( 3. \)
(c) The dimension of the row space of \( A \) is \( 3. \)
(d) The dimension of the null space of \( A \) is \( 4. \)
(e) The dimension of the null space of \( A^T \) is \( 3. \)
(f) Find a basis for the column space of \( A. \) ANS: Columns 1, 2, and 5 of \( A. \)
(g) Find a basis for the row space of \( A. \) ANS: The three non-zero rows of \( R. \)
(h) Find a basis for the null space of \( A. \)
ANS: Written as row vectors: \((3, -2, 1, 0, 0, 0, 0), (-5, 8, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0, 1, 0), (10, -4, 0, 0, -1, 0, 1)\)
(i) Express Column 7 of $A$ as a linear combination of the pivot columns of $A$.
ANS: column7 = $-10$ column1 + 4 column2 + column5.

(j) Use the information in the matrices $A$ and $R$ above to find the general (i.e, complete) solution to these two systems:
ANS: System 1: Written as a row vector: $x = (5, -8, 0) + c(3, -2, 1) c$ is any constant. System 2: There is no solution, because in the reduced form, the third equation is $0 = 1$.

21. Find a basis for the subspace of all vectors in $\mathbb{R}^4$ whose first coordinates are the sum of their third coordinates and twice their fourth coordinates. What is its dimension? ANS: dim 3; basis: $(1, 0, 1, 0)$ $(2, 0, 0, 1)$ $(0, 1, 0, 0)$

22. Are these functions $f_1(x) = 4x^3 - 5x^2 - x - 2$; $f_2(x) = x^3 + 3x^2 + 4x + 1$; $f_3(x) = 5x^3 - 19x^2 - 14x - 7$; $f_4(x) = x^3 + x^2 + 7x + 6$ independent or dependent? Find a basis for the subspace spanned by these functions.
Answer: $\begin{bmatrix} 4 & 1 & 5 & 1 \\ -5 & 3 & -19 & 1 \\ -1 & 4 & -14 & 7 \\ -2 & 1 & -7 & 6 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
Therefore $f_1, f_2$ and $f_4$ form a basis and the functions are dependent because $f_3 = 2f_1 - 3f_2$.

23. Write a basis for the subspace of all polynomials of degree 3 or less satisfying both $f(1) = f(2)$ and $f(-1) = f'(0)$. What is its dimension?
Let $f(x) = ax^3 + bx^2 + cx + d$. Equations are $a + b + c + d = 8a + 4b + 2c + d$ and $-a + b - c + d = c$.
$\begin{bmatrix} -7 & -3 & -1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix}$ reduces to $\begin{bmatrix} 1 & 0 & 7/10 & -3/10 \\ 0 & 1 & -13/10 & 7/10 \end{bmatrix}$ Therefore, $a = -7/10c + 3/10d, b = 13/10c - 7/10d$ so a basis is $-7/10x^3 + 13/10x^2 + x$ and $3/10x^3 - 7/10x^2 + 1 OR -7x^3 + 13x^2 + 10x$ and $3x^3 - 7x^2 + 10$
Dimension = 2.

In all the following assume $A$ and $B$ are $m$ by $n$ matrices, and assume $R$ is the reduced echelon form of $A$. Answer TRUE (T) if it is always true, and FALSE (F) otherwise:

1. Any independent set of $n$ vectors in $\mathbb{R}^n$ is a basis for $\mathbb{R}^n$.
2. Any set of vectors in $\mathbb{R}^n$ that spans $\mathbb{R}^n$ must be a basis for $\mathbb{R}^n$
3. The polynomials of degree $n$ or less is a vector space of dimension $n$.
4. The set of real-valued functions $g$ such that $g(2) = g(1)$ is a subspace of all real-valued functions $g$.
5. The set of $m$ by $n$ matrices is a vector space of dimension $n + m$.
6. The rank of $A^T$ is the same as the rank of $A$.
7. The reduced form of $A^T$ is $R^T$.
8. The null space of $A^T$ is the same as the null space of $A$.
9. The row space of $A^T$ is the same as the column space of $A$. 
10. The null space of \( R \) is the same as the null space of \( A \).
11. The null space of \( A \) is orthogonal to the column space of \( A \).
12. The orthogonal complement of a line through the origin in \( \mathbb{R}^3 \) is a plane.
13. The vectors orthogonal to \((1, -2)\) in \( \mathbb{R}^2 \) are on the line \( y = \frac{x}{2} \).
14. The orthogonal complement of the row space of \( A \) is the null space of \( A \).
15. The left null space of \( A \) is orthogonal to the column space of \( A \).
16. The column space of \( R \) is the same as the column space of \( A \).
17. The row space of \( R \) is the same as the row space of \( A \).
18. The pivot columns of \( R \) form a basis for the column space of \( A \).
19. The nonzero rows of \( R \) form a basis for the row space of \( A \).
20. The nonzero columns of \( A \) form a basis for the column space of \( A \).
21. For any given vector \( \bar{b} \), the set of solutions to \( A\bar{x} = \bar{b} \) is a subspace.
22. The set of vectors \( \{\bar{v}_1, \bar{v}_2, ..., \bar{v}_n\} \) is linearly independent if \( c_1\bar{v}_1 + ... + c_n\bar{v}_n = \bar{0} \) whenever \( c_1 = ... = c_n = 0 \).
23. Suppose that vector \( \bar{v} \) is orthogonal to the three vectors \( \bar{v}_1, \bar{v}_2, \) and \( \bar{v}_3 \). Then \( \bar{v} \) is orthogonal to the subspace spanned by \( \bar{v}_1, \bar{v}_2, \) and \( \bar{v}_3 \).
24. Suppose that \( \bar{v}_3 = 7\bar{v}_1 - \frac{3}{4}\bar{v}_4 \). Then \( \bar{v}_1, \bar{v}_2, \bar{v}_3 \) and \( \bar{v}_4 \) are dependent.
25. The set of vectors \( \{\bar{v}_1, \bar{v}_2, ..., \bar{v}_n\} \) spans a vector space \( V \) if every vector in \( V \) can be written as \( c_1\bar{v}_1 + c_2\bar{v}_2 + ... + c_n\bar{v}_n \).
26. The set of vectors \( \{\bar{v}_1, \bar{v}_2, ..., \bar{v}_n\} \) is linearly dependent if \( c_1\bar{v}_1 + c_2\bar{v}_2 + ... + c_n\bar{v}_n = \bar{0} \) has a solution.
27. The set of vectors \( \{\bar{v}_1, \bar{v}_2, ..., \bar{v}_n\} \) is a basis for a vector space \( V \) if every vector in \( V \) can be written as \( c_1\bar{v}_1 + c_2\bar{v}_2 + ... + c_n\bar{v}_n \) in one and only one way.
28. Any set of 8 vectors in \( \mathbb{R}^6 \) must be linearly dependent.
29. The dimension of a subspace is the number of vectors in it.
30. A subspace of \( \mathbb{R}^8 \) has dimension 8.
31. If a vector space \( V \) is spanned by a set of 7 vectors then \( V \) has dimension at least 7.
32. If a vector space \( V \) contains 7 independent vectors then \( V \) has dimension at most 7.
33. If \( S \) is a linearly independent subset of \( \mathbb{R}^9 \) with 9 elements, then \( S \) must span \( \mathbb{R}^9 \).
34. If \( S \) is a linearly independent set of vectors in \( \mathbb{R}^9 \) then \( S \) has at most 9 elements.
35. \( A\bar{x} \) is a linear combination of the columns of \( A \).
36. \( \bar{x}A \) is a linear combination of the rows of \( A \).
37. The row and column spaces of the reduced form of a matrix are the same as those of the original matrix.
38. The columns of a 20 by 19 matrix cannot be linearly independent.
39. Even if \( A \) is a singular \( n \) by \( n \) matrix, \( A\bar{x} = \bar{b} \) may still have a unique solution for some \( \bar{b} \) in \( \mathbb{R}^n \).
40. If a homogeneous system has more unknowns than equations, then it has an infinite number of solutions.