Math 330, Fall 08. Review of material since exam 2. Material below is primarily for review of what we have covered since the last exam. For review of material prior to exam #2, use old exams and exam reviews.

1. Diagonalize the matrix \[ A = \begin{bmatrix} 167 & -280 \\ 84 & -141 \end{bmatrix} \] and use this to find a cube root of \( A \) – that is, a matrix \( M \) such that \( M^3 = A \).

2. Diagonalize the matrix \[ A = \begin{bmatrix} 4 & -3 & 3 \\ 0 & 7 & -3 \\ 0 & 6 & -2 \end{bmatrix} \] Use this to find a square root of \( A \) – that is, a matrix \( M \) such that \( M^2 = A \). Can you find two different square roots of \( A \)? How would you do that?

3. Rupert claims that the matrix \[ A = \begin{bmatrix} 1 & 0 & 0 \\ -12 & -5 & 6 \\ -6 & -3 & 4 \end{bmatrix} \] has eigenvalues \(-2\) and \(1\). Find a basis for each of the corresponding eigenspaces.

4. Rupert claims that \(-6\) and \(7\) are eigenvalues of a matrix \( A \). When you row reduce \( A + 6I \) you get \[ \begin{bmatrix} 1 & 0 & 11 \\ 0 & 1 & -95 \\ 0 & 0 & 0 \end{bmatrix} \].

When you row reduce \( A - 7I \) you get \[ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]. Are Rupert’s answers correct? Explain.

5. Rupert claims that \((2, 5, 0), (1, 0, 0)\) and \((0, -1, 1)\) are eigenvectors of the matrix \[ A = \begin{bmatrix} 27 & -10 & -10 \\ 25 & -8 & -10 \\ 50 & -20 & -18 \end{bmatrix} \]. Determine if he is correct (without using determinants). Identify the eigenvalues corresponding to the correct eigenvectors.

6. Let \( S \) be the subspace of vectors \((x_1, x_2, x_3, x_4)\) in \( \mathbb{R}^4 \) for which \( x_1 = x_4 \) and \( x_2 + x_4 = 2x_3 \).

(a) Find a basis for \( S \).

(b) What is the matrix \( A \) in the formula for the projection matrix onto \( S \)?

(c) Which of these is the projection matrix onto \( S \)?
   (i) \((A^T A)^{-1} A^T\) (ii) \(A(A^T A)^{-1} A^T\) (iii) \((AA^T)^{-1} A^T\) (iv) \(A(AA^T)^{-1} A^T\)

7. Find the projection matrix onto the line through \((1, 3, -1)\) and the origin. Find the point on this line that is nearest to the point \((2, 1, 1)\).

8. Find the projection matrix onto the plane \( x = y \) in \( \mathbb{R}^3 \). Find the point on this plane that is nearest to the point \((5, 2, 3)\).

9. You want the best least squares fit of a cubic polynomial to the points \((-2, 1) \ (-1, 3) \ (0, 4) \ (2, 1) \ (4, 3)\).

(a) What is the matrix \( M \) in the formula for the best fit?

(b) What is the vector \( \bar{y} \) in the formula?

(c) Which of the following gives the coefficients for the best fit?
   (i) \((M^T M)^{-1} M^T \bar{y}\) (ii) \(M(M^T M)^{-1} M^T \bar{y}\) (iii) \((MM^T)^{-1} M^T \bar{y}\) (iv) \(M(MM^T)^{-1} M^T \bar{y}\)

(d) After applying the formula, you get \[ \begin{bmatrix} -1 \\ 0 \\ 7 \\ 11 \end{bmatrix} \]. What is the best cubic fit to the points?

(e) Now you want the best least squares fit of a straight line to the points. What is the matrix \( M \) in the formula?
10. Find the general solution to this system of differential equations:
\[
\frac{dx_1}{dt} = 23x_1 - 40x_2
\]
\[
\frac{dx_2}{dt} = 12x_1 - 21x_2
\]
(On the exam, your work should include all steps in diagonalizing the coefficient matrix and how the answer is obtained from this.)

11. Suppose \( A \) is a matrix with
\[
A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix};
A \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix};
A \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix}
\]
Find (a) \( A \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} \) (b) \( A \begin{bmatrix} 101 \\ 2 \\ 101 \end{bmatrix} \) (c) \( A^2 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \) (d) \( A \begin{bmatrix} 0 \\ 4 \\ 0 \end{bmatrix} \)

(e) Does \( A \) have an inverse?

There are two eigenvalues that can be determined from this information.

(f) One of the eigenvalues is \( \text{___________} \) and a corresponding eigenvector is \( \text{___________} \).

(g) The other eigenvalue is \( \text{___________} \) and a corresponding eigenvector is \( \text{___________} \).

(h) Is \( \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \) a linear combination of the columns of \( A \)? Explain your answer.

(i) Are the columns of \( A \) linearly independent?

(j) Can you ascertain what \( \det(A) \) is?

12. Suppose \( \lambda \) is an eigenvalue of the matrix \( A \). Prove that
(a) \( \lambda^2 \) is an eigenvalue of the matrix \( A^2 \) and (b) \( \frac{1}{\lambda} \) is an eigenvalue of the matrix \( A^{-1} \) if \( A \) is invertible.

13. Suppose \( A = SDS^{-1} \). Prove that (a) \( A^2 = SD^2S^{-1} \) (b) \( A^{-1} = SD^{-1}S^{-1} \) and (c) \( \det(A) = \det(D) \).

14. Prove that if \( A \) is a matrix and if \( (A^T A)\bar{x} = 0 \) then \( A\bar{x} = 0 \).

15. Find the determinant of the matrix \( M = \begin{bmatrix} 2 & 1 & 3 \\ -3 & 1 & 2 \\ 1 & -2 & 0 \end{bmatrix} \).

16. Suppose that \( \det \begin{bmatrix} 5 & 3 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & X & 0 & 1 \\ 1 & 2 & -1 & 1 \end{bmatrix} = 8 \). Find these:

(a) \( \det \begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & X & 0 & 1 \\ 5 & 3 & 1 & 0 \end{bmatrix} = \text{___________} \)

(b) \( \det \begin{bmatrix} 5 & 3 & 1 & 0 \\ -1 & 1 & 0 & -1 \\ 1 & X & 0 & 1 \\ 3 & 6 & -3 & 3 \end{bmatrix} = \text{___________} \)

(c) \( \det \begin{bmatrix} 5 & 3 & 1 & 0 \\ 4 & (3X - 1) & 0 & 4 \\ 1 & X & 0 & 1 \\ 1 & 2 & -1 & 1 \end{bmatrix} = \text{___________} \).

(d) \( \det \begin{bmatrix} 5 & 3 & 1 & 0 \\ -2 & (1 - X) & 0 & -2 \\ 1 & X & 0 & 1 \\ 1 & 2 & -1 & 1 \end{bmatrix} = \text{___________} \).
In the following assume this: Let \( A, B \) and \( C \) be 6 by 6 matrices. Assume \( \det(A) = 3, \det(B) = 2, \det(C) = 0. \) Answer True if it is always true, and False otherwise.

1. \( \det(3A) = 9 \)
2. \( \det(A^3) = 18 \)
3. \( \det(B^{-1}) = 1/2 \)
4. \( \det(-B) = -2 \)
5. \( \det(A + B) = 5 \)
6. \( \det(C + B) = 2 \)
7. \( \det(5C) = 0 \)
8. \( \det(AB) = 6 \)
9. \( A \) must be invertible.
10. \( C \) is non-singular.
11. The rows of \( C \) must be linearly dependent.
12. The columns of \( A \) must be linearly independent.
13. \( C \) must have either a row or a column of zeros.
14. If \( M \) is obtained from \( A \) by adding the first row to the fourth row then \( \det(M) = 6. \)
15. If \( M \) is obtained from \( A \) by multiplying the fourth row by 4 then \( \det(M) = 12. \)
16. If \( M \) is obtained from \( A \) by replacing the first row by the fourth row then \( \det(M) = 3. \)
17. If \( M \) is obtained from \( A \) by interchanging the first and fourth rows then \( \det(M) = -3. \)
18. If \( \lambda \) is an eigenvalue of the matrix \( B \) then \( \lambda \) is also an eigenvalue of the matrix \( B^{-1}. \)
19. 2 must be an eigenvalue of the matrix \( B. \)
20. 0 must be an eigenvalue of the matrix \( C. \)
21. \( \vec{0} \) must be an eigenvector of the matrix \( C. \)
22. From what is given, one can’t tell whether 0 is an eigenvalue of the matrix \( A \) or not.

In the following assume this: Let \( S \) be a subspace of \( \mathbb{R}^6 \) and assume \( \vec{b} \) is a point in \( \mathbb{R}^6 \) not in \( S. \) Suppose the vectors \( \vec{v}_1, \vec{v}_2, ..., \vec{v}_n \) form a basis for \( S \) and suppose the \( \vec{p} \) is the point in \( S \) nearest to \( \vec{b}. \) Let \( A \) be the matrix in the formula for the projection of \( \vec{b} \) onto \( S. \) Answer True if it is always true, and False otherwise.

1. The formula for \( \vec{p} \) is \( A(A^T A)^{-1} A^T \vec{b}. \)
2. The projection matrix onto \( S \) is \( A. \)
3. \( \vec{b} \) is orthogonal to \( \vec{p}. \)
4. \( \vec{b} - \vec{p} \) is orthogonal to \( \vec{p}. \)
5. \( (\vec{p} - \vec{b}) \cdot \vec{v}_i = 0 \) for each \( i = 1, ..., n. \)
6. \( \vec{p} \) is orthogonal to \( \vec{v}_i \) for each \( i = 1, ..., n. \)
7. \( \vec{b} - \vec{p} \) is orthogonal to each \( \vec{s} \) in \( S. \)
8. \( \vec{b} \cdot \vec{s} = \vec{p} \cdot \vec{s} \) for each \( \vec{s} \) in \( S. \)
9. \( \vec{b} \) is orthogonal to each \( \vec{s} \) in \( S. \)
10. $\bar{p} - \bar{b}$ is in the null space of $A^T$.
11. $n$ cannot equal 6.
12. $A$ has 6 rows.
13. $A$ is $n$ by 6.
14. $S$ is actually the column space of $A$.
15. The rows of the matrix $A$ are the basis vectors $\bar{v}_1, \bar{v}_2, ..., \bar{v}_n$.
16. There must be a vector $\bar{c}$ for which $A\bar{c} = \bar{p}$.
17. There must be a vector $\bar{c}$ for which $A\bar{c} = \bar{b}$.
18. $\bar{b}$ is a linear combination of $\bar{v}_1, \bar{v}_2, ..., \bar{v}_n$.
19. There must be numbers $a_1, ..., a_n$ so that $\bar{p} = a_1\bar{v}_1 + a_2\bar{v}_2 + ... + a_n\bar{v}_n$.
20. The null space of $A^TA$ is exactly the same as the null space of $A$, even though $A$ may not be square.
21. $\left(A^TA\right)^{-1} = A^{-1}$.
22. $\left(A^TA\right)^{-1} = A^{-1}(A^T)^{-1}$.

In the following assume this: Let $(x_1, y_1), ..., (x_5, y_5)$ be five given points with no two of the $x$ values the same. Let $\bar{Y} = (y_1, ..., y_5)$. Let $M$ be the matrix in the formula for the coefficients of the best least squares fit of a polynomial of degree $n$ and let $f(x)$ denote that polynomial. Answer True if it is always true, and False otherwise.

1. True
2. True
3. True
4. False
5. True
6. True
7. True
8. True
9. True
10. True
11. True
12. True
13. True
14. False
15. True

Statisticians call the best least squares fit of a line to the points the **regression line**.