(1) **P.100, 3.** Show that if $A$ is closed in $X$ and $B$ is closed in $Y$, then $A \times B$ is closed in $X \times Y$.

Note that $X \times Y - A \times B = ((X - A) \times Y) \cup (X \times (Y - B))$, the two sets on the right are each open (in fact, basis elements), and union of open sets is open.

Proof that $X \times Y - A \times B = ((X - A) \times Y) \cup (X \times (Y - B))$:

$(x,y) \in X \times Y - A \times B$ ⇔ $(x,y) \notin A \times B$
⇔ $x \notin A$ or $y \notin B$
⇔ $x \in X - A$ or $y \in Y - B$
⇔ $(x,y) \in ((X - A) \times Y) \cup (X \times (Y - B))$.

(2) **Page 100, 6c.** Prove that $\bigcup \overline{A_\alpha} \supset \bigcup \bar{A}_\alpha$; give an example where equality fails.

Note that $x \in \bigcup \bar{A}_\alpha \iff x \in \bar{A}_{\alpha_0}$ for some $\alpha_0 \implies$

each open set containing $x$ intersects $A_{\alpha_0}$ for some $\alpha_0$ (Theorem 17.5) \implies
each open set containing $x$ intersects $\bigcup A_\alpha \implies x \in \bigcup \overline{A_\alpha}$.

Example: Let $A_n = \left[\frac{1}{n}, 2\right]$, $n \in \mathbb{N}$.
Then $\bigcup A_n = (0, 2]$, $\bigcup \overline{A_n} = [0, 2]$, whereas $\bar{A}_n = A_n$ and $\bigcup \overline{A_n} = (0, 2] \neq \bigcup \overline{A_n}$.

(3) **Page 100, 8a.** Prove or disprove that $\overline{A \cap B} = \bar{A} \cap \bar{B}$.

$x \in \overline{A \cap B}$
⇔ each open set containing $x$ intersects $A \cap B$ (Theorem 17.5)
⇒ each open set containing $x$ intersects $A$ as well as $B$
⇔ $x \in \bar{A}$ and $x \in \bar{B}$
⇔ $x \in \bar{A} \cap \bar{B}$. (Theorem 17.5)

So we have $\bar{A} \cap \bar{B} \subset \bar{A} \cap \bar{B}$.

Note that one of the implications above is in only one direction. Even if every open set $U$ containing $x$ intersects $A$ as well as $B$, the intersection itself could still be two different sets, and we may have $U \cap (A \cap B)$ an empty set. This is illustrated in the example below:

Consider the subsets $A = (0, 1)$ and $B = (1, 2)$ of $\mathbb{R}$. Then $\bar{A} = [0, 1]$, $\bar{B} = [1, 2]$, $\bar{A} \cap \bar{B} = \{1\}$, but $A \cap B = \emptyset = \bar{A} \cap \bar{B}$.

(4) **Page 111, 1.** Let $f: \mathbb{R} \to \mathbb{R}$. Assume that for each $a \in \mathbb{R}$ and $\epsilon > 0$, there exists a $\delta > 0$ (which typically depends on $\epsilon$ and $a$) such that

$$|x - a| < \delta \implies |f(x) - f(a)| < \epsilon.$$ 

We want to show that for each open set $V \subset \mathbb{R}$, the set $f^{-1}(V)$ is open in $\mathbb{R}$. 


Let $f$ be as above and let $V$ be an open set in $\mathbb{R}$. Note that the hypothesis on $f$ may be rewritten as:

For each $a \in \mathbb{R}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq f^{-1}((f(a) - \epsilon, f(a) + \epsilon)$.

If $f^{-1}(V)$ is empty, as the empty set is open, there is nothing to prove. Otherwise $f^{-1}(V)$ is not empty. It suffices to show that for each $a \in f^{-1}(V)$, there is an open interval containing $a$ that is inside $f^{-1}(V)$.

Let $a \in f^{-1}(V)$. Clearly, $f(a) \in V$ and since $V$ is open, by the definition of the standard topology on $\mathbb{R}$, there is an open interval, say $(p, q)$ such that $f(a) \in (p, q) \subseteq V$.

Let $\epsilon = \min\{f(a) - p, q - f(a)\}$. Then $(f(a) - \epsilon, f(a) + \epsilon) \subseteq V$.

It follows that $f^{-1}((f(a) - \epsilon, f(a) + \epsilon)) \subseteq f^{-1}(V)$. Now, by hypothesis, there is a $\delta > 0$ such that $(a - \delta, a + \delta) \subseteq f^{-1}((f(a) - \epsilon, f(a) + \epsilon)) \subseteq f^{-1}(V)$.

(5) **Page 111, 2.** Suppose that $f : X \to Y$ is continuous. If $x$ is a limit point of the set $A$, is it necessarily true that $f(x)$ is a limit point of the set $f(A)$?

**Answer:** No. Example: Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = 2$. Then the point 1 is a limit point of the set $A = [0, 1]$, but the set $f(A) = \{2\}$ has no limit points.

(6) **Page 111, 5.** The function $f(x) = bx + a$ gives a homeomorphism between $(0, 1)$ and $(a, b)$ as well as between $[0, 1]$ and $[a, b]$.

(7) **P.100, 1.** Use that (i) $X$ and $\phi$ are complements of sets in $\mathcal{C}$.

(ii) Complement of a finite union is a finite intersection of complements.

(iii) Complement of an arbitrary intersection is the union of complements.

(8) **Page 111, 12.**

$$F(x, y) = \begin{cases} xy/(x^2 + y^2) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

(a) If $y = 0$, $F(x, y) = 0$, which is a constant function in $x$, therefore continuous.

If $y$ is a fixed nonzero number, $F(\ , y)$ is a rational function in $x$ with nonzero denominator, so it is continuous on $\mathbb{R}$.

Similarly, for a fixed $x$, $F(x, \ )$ is a continuous function on $\mathbb{R}$.

(b) $g(x) = F(x, x) = \begin{cases} 1/2 & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

This is not a continuous function because the interval $(-0.1, 0.1)$ is open in $\mathbb{R}$, but its preimage $g^{-1}(-0.1, 0.1) = \{0\}$ is not.

(c) It is easy to see that the function $h(x) = (x, x)$ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^2$. If $F$ was continuous, then the composition $g = F \circ h$ would be continuous, contradicting that $g$ isn’t.