Math 440, Quiz Solution, March 28, 2012
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Justify each answer.

(1) Let $X = \{a, b, c, d\}$, $Y = \{p, q, r\}$, and let $T_X = \{\emptyset, X, \{a\}, \{b, c\}\}$, and $T_Y = \{\emptyset, Y, \{q\}, \{r\}\}$ be the topologies on $X$ and $Y$ respectively. Assume the product topology on $X \times Y$.

(a) Is the set $\{(a, q), (b, r), (c, r)\}$ an open subset of $X \times Y$? 2pts

Yes, $\{(a, q), (b, r), (c, r)\} = (\{a\} \times \{q\}) \cup (\{b, c\} \times \{r\})$.

The sets $\{a\}$, $\{b, c\}$ are open in $X$; $\{q\}$, $\{r\}$ are open in $Y$. Union of products of open sets is open in the product space.

(b) Let the function $f: X \to Y$ be given by $f(x) = \begin{cases} \quad q, & x = a, b, d, \\
\quad r, & \text{otherwise.} \end{cases}$

Is this function continuous? 2pts

No, $\{q\}$ is open in $Y$, but $f^{-1}\{q\} = \{a, b, d\}$ which is not open in $X$.

(c) Is the function $g: X \times Y \to Y$ given by $g(x, y) = p$ for all $(x, y) \in X \times Y$ continuous? 1pt

Yes, it is a constant function. $g^{-1}(U) = \emptyset$ if $p \notin U$, and $g^{-1}(U) = X \times Y$ if $p \in U$.

(d) Is either $X$ or $Y$ metrizable? 2pts

No, neither has the discrete topology. See (2) below.

(2) Show that if a finite topological space is metrizable, then the topology is the discrete topology. 3pts

Let $X = \{x_1, x_2, \ldots, x_n\}$ with metric $d$.
Let $r = \min \{d(x_i, x_j) \mid i \neq j\}$. Then $B(x, r) = \{x\}$.

As singleton sets are open, any subset if $X$ is open.