Algebraic Topology Homework on Section 1

1. Show that the barycentric coordinates \( t_i \) are continuous functions on the simplex \( \sigma \) spanned by \( a_0, \ldots, a_n \).

**Solution** Let \( B : \mathbb{R}^N \rightarrow \mathbb{R}^N \) be the “shift” or translation defined by \( B(v) = v - a_0 \). Let \( A \) be the matrix with columns \( a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0, b_1, \ldots, b_{N-n} \), where \( b_i \) are chosen to complete the linearly independent set \( \{a_1 - a_0, \ldots, a_n - a_0\} \) to make a basis.

As a linear transformation, \( A \) sends the standard basis \( e_1 = (1, 0, \ldots, 0), \ldots, e_N = (0, \ldots, 0, 1) \) to its column vectors. \( A^{-1} \) therefore takes the column vectors of \( A \) to the standard basis elements.

Define \( T \) by \( T = A^{-1} \circ B \) (compose these maps from \( \mathbb{R}^N \) to itself). Note that although \( T \) is “linear” in the sense of analysis (it equals its first order Taylor approximation) not linear in the linear algebra sense (it does not take zero to zero, because of the shift). This is analogous to the map \( f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 2x + 3 \), which is not linear since \( f(x_1 + x_2) \neq f(x_1) + f(x_2) \). We’ll reserve the term linear for the linear algebra type of linearity, and use the term **affine** to describe a map which is linear up to a shift (i.e. addition of a constant).

**Property (1)** The map \( B \) is clearly continuous. Any linear map is continuous, so \( T = A^{-1} \circ B \) is continuous.

\[
T \left( \sum_{i=0}^{n} t_i a_i \right) = A^{-1} \left( \left( \sum_{i=0}^{n} t_i a_i \right) - a_0 \right) = A^{-1} \left( \left( \sum_{i=0}^{n} t_i a_i \right) - \left( \sum_{i=0}^{n} t_i a_0 \right) \right) = \sum_{i=0}^{n} A^{-1} t_i (a_i - a_0) = \sum_{i=0}^{n} t_i A^{-1} (a_i - a_0) = \sum_{i=0}^{n} t_i e_i.
\]

Since the barycentric coordinates \( t_i \) are the (first \( n \)) components of this continuous map \( T \), they are continuous.

**Property (2)** Show that \( \sigma \) is the union of all line segments from \( a_0 \) to points on the simplex spanned by \( a_1, \ldots, a_n \), and that two such segments intersect only at \( a_0 \).

**Solution** Let \( s_1, \ldots, s_n \) be real numbers with \( s_i \geq 0 \) and \( \sum_{i=1}^{n} s_i = 1 \). Then \( \sum_{i=1}^{n} s_i a_i \) is a point on the simplex spanned by \( a_1, \ldots, a_n \). The line segment \( \{t_0 a_0 + (1-t_0) \sum_{i=1}^{n} s_i a_i | 0 \leq t_0 \leq 1 \} \) is in \( \sigma \), because \( t_0 s_i \geq 0 \) and \( t_0 + (1-t_0)(s_1 + \cdots + s_n) = 1 \). This shows the union of the line segments is in \( \sigma \).

Given a point \( \sum_{i=0}^{n} t_i a_i \) in the simplex with \( t_0 \neq 1 \), set \( s_i = \frac{t_i}{1-t_0} \) for \( i = 1, \ldots, n \). This shows that all of the simplex spanned by \( a_0, \ldots, a_n \) except \( a_0 \) is in the union of the line
segments. Clearly, \(a_0\) is also in the union (it is in each line segment). Therefore \(\sigma\) is in the union of the line segments, so the two are equal.

Finally, we must show that two line segments joining points in the simplex spanned by \(a_1, \ldots, a_n\) to \(a_0\) intersect only at \(a_0\). It is clear that any pair of these line segments intersect at \(a_0\). If two segments intersect at another point, then they must be part of the same line, since two points determine a line. The other endpoints of these segments (other than \(a_0\)) are two points in the \(n-1\) dimensional plane containing \(a_1, \ldots, a_n\). But then the line meets this plane in two points so it is contained in the plane. This contradicts the fact that \(a_0\) is not in this plane (since \(a_0, \ldots, a_n\) are geometrically independent.

**Property (3)** Show that \(\sigma\) is a compact convex subset of \(\mathbb{R}^N\), and that \(\sigma\) is the intersection of all convex sets containing \(a_0, a_1, \ldots, a_n\).

**Solution** Since \(\sigma \subset \mathbb{R}^N\), we can show compactness by showing closedness and boundedness. Boundedness is easy, since \(\|\sum_{i=0}^{n} t_i a_i\| \leq \max\{t_0, \ldots, t_n\} \max\{|\|a_0\|, \ldots, |\|a_n\|\}\) \leq \max\{|\|a_0\|, \ldots, |\|a_n\|\} \).

Since all the \(t_i\) functions are continuous and they extend to continuous functions from \(\mathbb{R}^N\), so is the map \(\tilde{t} = (t_0, \ldots, t_n) : \sigma \rightarrow \mathbb{R}^{n+1}\) and its extension to \(\mathbb{R}^N\). Since all the inequalities \(t_i \geq 0\) and \(t_0 + \cdots + t_n = 1\) determine closed subsets of \(\mathbb{R}^{n+1}\), the set \(S\) of solutions to this collection (the intersections of the solution sets for each one) is a closed set in \(\mathbb{R}^N\). It follows that \(\sigma\), the inverse image of this closed set under \(\tilde{t}\), is closed.

For convexity, suppose that \((t_0, \ldots, t_n)\) and \((t'_0, \ldots, t'_n)\) are two distinct \(n+1\)-tuples satisfying the nonnegativity conditions and the “sum equals one” condition. Then \(t_0, \ldots, t_n + (1-t) t'_0, \ldots, t'_n\) parameterizes a line segment between these two points, as \(0 \leq t \leq 1\). The corresponding curve \(t \sum_{i=0}^{n} t_i a_i + (1-t) \sum_{i=0}^{n} t'_i a_i = \sum_{i=0}^{n} (t t_i + (1-t) t'_i) a_i\) is a line segment connecting \(\sum_{i=0}^{n} t_i a_i\) to \(\sum_{i=0}^{n} t'_i a_i\). Since conditions on \(t_i\), \(t_i\) and \(t\) imply that \(t t_i + (1-t) t'_i\) are all nonnegative and sum to 1. This proves \(\sigma\) is convex.

Finally, suppose that \(S\) is a convex set containing \(a_0, \ldots, a_n\). Then for every point \(x = \sum_{i=0}^{n} t_i a_i \in \sigma\), we can show \(x \in S\) as follows. The point \(b_1 = \frac{t_0}{t_0 + t_1 + \cdots + t_n} a_0 + \frac{t_1}{t_0 + t_1 + \cdots + t_n} a_1\) must be in \(S\), since \(a_0\) and \(a_1\) are in \(S\), by convexity. Thus

\[
\begin{align*}
b_2 &= \frac{t_0 + t_1}{t_0 + t_1 + t_2} b_1 + \frac{t_2}{t_0 + t_1 + t_2} b_2 \\
&= \frac{t_0}{t_0 + t_1 + t_2} a_0 + \frac{t_2}{t_0 + t_1 + t_2} a_2
\end{align*}
\]

is also in \(S\), by convexity. By an easy induction argument, we can continue until we get to

\[
\frac{t_0}{t_0 + \cdots + t_n} a_0 + \cdots + \frac{t_n}{t_0 + \cdots + t_n} a_n \in S,
\]

but \(t_0 + \cdots + t_n = 1\), so \(x \in S\).

It follows that \(\sigma\) is contained in every convex set containing \(\{a_0, \ldots, a_n\}\). Since we showed \(\sigma\) is convex, the intersection of all such convex sets can be no larger than \(\sigma\). It must be exactly \(\sigma\).
2. **Property (4)** Show that any simplex has exactly one geometrically independent set of points spanning it. In other words, the vertices are uniquely determined (as a set) by the simplex.

**Solution** We first show that any point \( x \in \sigma \) which is not one of the vertices \( a_0, \ldots, a_n \) lies on an open line segment in \( \sigma \).

Suppose \( x = \sum_{i=0}^{n} t_i a_i \) is not a vertex. Then no \( t_i \) equals 1. It follows that at least two of the \( t_i \) are nonzero, say \( t_0 \) and \( t_1 \) without any loss of generality. Choose \( \epsilon > 0 \) so that \( (t_0 - \epsilon, t_0 + \epsilon) \subset (0, 1) \) and likewise for \( t_1 \). Then for \( t \in (-\epsilon, \epsilon) \), \( (t_0 + t)a_0 + (t_1 - t)a_1 + \sum_{i=2}^{n} t_i a_i \) is in \( \sigma \). It follows that every point in \( \sigma \) which is not one of its vertices is on an open line segment contained in \( \sigma \).

Next we'll show that the vertices are not on such open line segments. If any vertex, say \( a_0 \), were on an open segment contained in \( \sigma \), then there would exist points \( x, y \in \sigma \) with \( a_0 = tx + (1-t)y \). If \( x = \sum_{i=0}^{n} t_i a_i \) and \( y = \sum_{i=0}^{n} s_i a_i \), then this means \( a_0 = \sum_{i=0}^{n} (tt_i + (1-t)s_i) a_i \), this violates the geometric independence of \( a_0, \ldots, a_n \), unless \( t_0 = s_0 = 1 \) and the others are zero, in which case \( x = y \). Thus \( a_0 \) is not on an open line segment.

3. **Property (5)** Show that \( \text{Int}(\sigma) \) is convex and is open in the plane \( P \); its closure is \( \sigma \). Furthermore \( \text{Int}(\sigma) \) equals the union of all open line segments joining \( a_0 \) to points of \( \text{Int}(s) \), where \( s \) is the face of \( \sigma \) opposite \( a_0 \).

**Solution** Note that \( \text{Int}(\sigma) \) consists of points \( x \) of \( \sigma \) for which all barycentric coordinates \( t_i(x) \) are positive.

The functions \( t_i \) are continuous functions from the plane \( P \) spanned by vertices of \( \sigma \) to \( \mathbb{R} \). The preimage of the open interval \( (0, \infty) \) under each \( t_i \) is therefore open in \( P \). \( \text{Int}(\sigma) \) is the intersection of these finitely many open sets and therefore it is open.

\( \text{Int}(\sigma) \) is easily seen to be convex as the convex combination \( (1-t)x + ty \) of any \( x, y \in \text{Int}(\sigma) \) is in \( \text{Int}(\sigma) \).

\[(1-t)x + ty = (1-t) \sum_{i=0}^{n} t_i a_i + t \sum_{i=0}^{n} s_i a_i = \sum_{i=0}^{n} ((1-t)t_i + ts_i) a_i \] and if \( t_i \) and \( s_i \) are positive, so is \( (1-t)t_i + ts_i \).

As \( \sigma \) is closed, the closure of \( \text{Int}(\sigma) \) is a subset of \( \sigma \). On the other hand if a point \( x \) in \( \text{Bd}(\sigma) \) is not in the closure of \( \text{Int}(\sigma) \), one can easily construct a sequence of points in \( \text{Int}(\sigma) \) which converges to \( x \).

Without loss of generality, assume that \( x = \sum_{i=0}^{n} t_i a_i \), and \( t_k(x) > 0 \) for \( 0 \leq i \leq k \) and \( t_{k+1}(x) = \cdots = t_n(x) = 0 \).

Let \( r = \min\{t_i(x) | 0 \leq i \leq k \}/2 \). Let \( x_m = \sum_{i=0}^{k} (t_i(x) - r/m^{(k+1)})a_i + \sum_{i=k+1}^{n} (r/m^{(n-k)}) \). Then \( x_m \in \text{Int}(\sigma) \) and the sequence converges to \( x \).

Finally, we show that \( \text{Int}(\sigma) \) equals the union of open line segments which join a vertex \( a_0 \) to points in the interior of the opposite face. Let \( x = \sum_{i=0}^{n} t_i a_i \), \( t_i > 0 \), \( \sigma t_i = 1 \). Rewrite this as \( x = t_0 a_0 + (1-t_0) \sum_{i=1}^{n} \frac{t_i}{1-t_0} a_i \).