Homework on Section 22 of TOPOLOGY

3. Let \( A = \{ x \times y \mid x \geq 0 \text{ or } y = 0 \} \subset \mathbb{R}^2 \). Show that the projection map onto the first coordinate, restricted to \( A \), is a quotient map.

Solution Call the restriction of the projection map \( q \). It is easy to see that \( q \) is onto.

Consider \( (a, b) \subset \mathbb{R} \).

\( q^{-1}(a, b) = ((a, b) \times \mathbb{R}) \cap A \) which is clearly open. As (i) an open set in \( \mathbb{R} \) is a union of open intervals, (ii) \( q^{-1}(U) = \bigcup q^{-1}(V_a) \) and (iii) union of open sets is open, it follows that preimage of an open set is open. So \( q \) is continuous.

We want to show that if \( q^{-1}(U) \) is open in \( A \) for some \( U \subset \mathbb{R} \), then \( U \) is open.

Clearly, \( U \subset (-\infty, 0) \) is open in \( \mathbb{R} \) if and only if \( q^{-1}(U) = U \times \{0\} \) is open in \( A \).

For \( U \subset (0, \infty) \), we have \( q^{-1}(U) = U \times \mathbb{R} \) which is open in \( A \) if and only if \( U \) is open in \( \mathbb{R} \).

Now suppose we have a set \( U \subset \mathbb{R} \) such that \( 0 \in U \) and \( q^{-1}(U) \) is open in \( A \).

Then \( q^{-1}(U) \cap \{ x \times y \mid y = 0, x < 0 \} \) is open and from the above discussion the image of this, which equals \( U \cap (-\infty, 0) \), is open.

Similarly, \( q^{-1}(U) \cap \{ x \times y \mid x > 0 \} \) is open and so it follows that \( U \cap (0, \infty) \) is open.

Finally, there exists a basis element \( (a, b) \times (c, d) \) of the product topology on \( \mathbb{R}^2 \) such that \( 0 \times 0 \in (a, b) \times (c, d) \subset q^{-1}(U) \). It follows that \( 0 \in (a, b) \subset U \).

Thus \( U \) is the union of three open sets and therefore it is open.

The map \( q \) is not open. The set \( V = \{ x \times y \mid y > 3 \} \cap A \) is an open set in \( A \). Its image \([0, \infty)\) is not open in \( \mathbb{R} \).

The map \( q \) is not closed. The set \( F = \{ x \times y \mid y = \frac{1}{x}, x > 0 \} \) is a closed subset of \( A \). Its image \((0, \infty)\) is not a closed subset of \( \mathbb{R} \).

4a. Define an equivalence relation on \( X = \mathbb{R}^2 \) as

\[
x_0 \times y_0 \sim x_1 \times y_1 \text{ if } x_0 + y_0^2 = x_1 + y_1^2.
\]

Let \( X^* \) be the set of equivalence classes with \( U \subset X^* \) defined to be open if and only if \( p^{-1}(U) \subset \mathbb{R}^2 \) is open, where \( p \) is the map that carries a point in \( \mathbb{R}^2 \) to its equivalence class. (See the definition and discussion on page 138 and the definition on page 139.) Which familiar space is \( X^* \) homeomorphic to?

Solution We claim that \( X^* \) is homeomorphic to \( \mathbb{R} \) with the standard topology generated by open intervals and a homeomorphism is given by mapping the equivalence class of an element \( a \times b \) of \( \mathbb{R}^2 \) to \( a + b' \in \mathbb{R} \).

Let \( g([a \times b]) = g([a' \times b']) \). Then \( a + b^2 = a' + b'^2 \) and therefore \( a \times b \sim a' \times b' \). It follows that \( [a \times b] = [a' \times b'] \). The map \( g \) is one-to-one.

The map \( g \) is onto, as given \( r \in \mathbb{R} \), \( g \) maps the equivalence class of \( r \times 0 \) to \( r \).

For continuity of \( g \), we note that

\[
p^{-1}g^{-1}(c, d) = \{ x \times y \in \mathbb{R}^2 \mid c < x + y^2 < d \},
\]

where \( (c, d) \subset \mathbb{R} \) is an open interval. It is easy to see that this preimage in \( \mathbb{R}^2 \) is open. It follows that \( g \) is continuous.

Finally, let \( U \subset X^* \) be an open set. We want to show that \( g(U) \subset \mathbb{R} \) is open.

Let \( r \in g(U) \). Then since \( g(r \times 0) = r \), and \( g \) is one-to-one, it follows that \([r \times 0] \in U \) or equivalently, \( r \times 0 \in p^{-1}(U) \).

Since \( U \) is open, by the definition of the topology on \( X^* \), \( p^{-1}(U) \) is open in \( \mathbb{R}^2 \). So there exists intervals \( (c, d) \) and \( (e, f) \) such that

\[
r \times 0 \in (c, d) \times (e, f) \subset p^{-1}(U).
\]

Clearly, we have \( c < r < d \) and \( e < f \). So if \( c < x < d \) then \( x \times 0 \in p^{-1}(U) \Leftrightarrow [x \times 0] \in U \Rightarrow x = x + 0^2 \in g(U) \). It follows that \( r \in (c, d) \subset g(U) \). So \( g(U) \) is open.

We have shown that \( g \) is a homeomorphism.