Ambiguity, pessimism, and rational religious choice

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Published online: 8 July 2009
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Abstract Using a subclass of the $\alpha$-maximin expected-utility preference model, in which the decision maker’s degree of ambiguity and degree of pessimism are each parameterized, we present a theory of religious choice in the Pascalian decision theory tradition, one that can resolve dilemmas, address the “many Gods objection,” and address the ambiguity inherent in religious choice. Parameterizing both the degree of ambiguity and the degree of pessimism allows one to examine how the two interact to impact choice, which is useful regardless of the application. Applying this model to religious choice is a move beyond subjective expected-utility theory, allowing us to show that a change in either the degree of ambiguity or the degree of pessimism can lead a decision maker to “convert” from one religion to another.

Keywords Choice of religion · $\alpha$-maximin expected utility

…I know…I must soon die, but what I know least is the very death I cannot escape.—Pascal (1670[1958], fragment 194)

1 Introduction

Suffering from ill health most of his adult life, noted scientist and mathematician Blaise Pascal experienced what he said knew least on August 16, 1662 at age 39. He died before completing the comprehensive apology for the Christian faith he was working on, having shifted his efforts away from math and science because of a “second
conversion” after a moving personal experience in 1654. Pascal’s Pensées (Pascal 1670[1958]), a collection of Pascal’s thoughts on religion and philosophy, was published posthumously. Among these thoughts, one finds an innovative application of logic to religious choice, thinking that has been labeled “the advent of decision theory” (Jorden 1994a, p. 3).

It may not be incidental that thinking about life after death played a role in the development of decision theory. What happens at the moment of death is perhaps the quintessential example of uncertainty, and because this uncertainty makes people anxious, it is worth examining. Smith (1759[2000], Part I, paragraph I.I.13) called the “dread of death” the “great poison to human happiness.”

In its simplest form, Pascal’s “wager” can be interpreted as a choice between theism and atheism. In Pascal’s words:

God is or He is not. But to which side will we incline? …What will you wager? … You must wager. It is not optional … Let us weigh the gain and the loss in wagering that God is. … If you win, you win everything, if you lose you lose nothing. Do not hesitate then; wager that he does exist.—Pascal (1670[1958], fragment 233)

We see in Pascal’s words the elements of modern decision theory. The state space representing uncertainty is parsed into mutually exclusive events: “God is” and “God is not.” Alternative acts are recognized: “wager that God is” and “wager that God is not.” A consequence is associated with each action–state combination. While Pascal does not make these consequences explicit, he implicitly assumes one decision alternative “weakly dominates” the other. That is, in this pioneering decision theory problem, the weak dominance concept was applied to offer an explanation for why it is rational for people to choose theism over atheism.

Of course, this simple normative model of religious choice is open to many criticisms. One is that weak dominance may not apply. Indeed, the choice becomes a dilemma if there are costs associated with gambling on God that exceed the costs associated with the alternative, for the best choice depends on which state is “true.”

Second, the “many Gods objection” to Pascal’s wager is the objection to recognizing

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1 Pascal’s “first conversion” was a 1646 commitment to Jansenism, a relatively Puritanical and relatively Augustinian practice of Roman Catholicism that conformed to the writings of Cornelius Jansen. The “second conversion” is associated with an event reportedly occurring on November 23, 1654. Pascal and some friends were riding in a carriage when the team of horses pulling it plunged off a bridge. Because the reins broke, the carriage did not follow, but was left half on and half off the bridge. Pascal apparently fainted out of a fear of the nearness of death, and was unconscious for some time. Later, while recovering, Pascal had what he described as an intense religious vision. The notes he immediately took down to remind himself of the vision, known as the Memorial, were inadvertently found by a servant after his death, sewn into Pascal’s coat.

2 McClellan (1994, p. 118) discusses this and other criticisms.

3 See Jorden (1994b) for a discussion of the many Gods objection. Hacking (1994) and Morris (1994) suggest Pascal did not intend for us to partition the set of choices into theism and atheism. Rather, it is more accurate to say Pascal’s intent was to make the partition “pursuing God” with actions that will probably lead one to believe versus “not bothering about such things” (Hacking 1994, p. 25). Similarly, Morris (1994, p. 56) contends Pascal’s intent was to use his wager to put the unbeliever on the path to belief in the Christian faith which Pascal had adopted. However, the many Gods objection remains as long as one admits the possibility that all do not find the same religion in the search.
only two religious alternatives. Finally, and most significantly for this article, the beliefs of decision makers may be ambiguous to varying degrees, and the degree of pessimism may vary; these differences across decision makers may explain differences in religious choice.

In this article, we use recent developments in decision theory to extend the Pascalian model of religious choice. In particular, we develop and apply a version of the $\alpha$-maximin expected-utility ($\alpha$-MMEU) model.\footnote{See, e.g., Cohen (1992); Jaffray and Philippe (1997); Ghirardato et al. (1998); Ghirardato et al. (2004); Siniscalchi (2006); Ludwig and Zimper (2006); Olszewski (2007); Eichberger et al. (2008).} Our contribution is twofold. First, by using a specification of $\alpha$-MMEU model where the degree of ambiguity is represented by a single parameter, we are able to examine how choice is affected as the degrees of ambiguity and pessimism interact. The interactions we delineate apply to uncertain choice in general, not just religious choice, so the reader may find the model of interest for some other application. Second, our model provides a normative theory of religious choice that applies to a broad class of decision makers, who vary in the degree to which beliefs are ambiguous and vary in the degree to which they are pessimistic.

While SEU theory can resolve dilemmas and address the many gods objection, it assumes the decision maker’s religious beliefs are unambiguous, and can be represented by a unique probability measure. In this SEU environment, the degree of pessimism, which is equivalent to the degree of ambiguity aversion, cannot impact choice. The more general $\alpha$-MMEU model includes the SEU model as a special case, but allows for ambiguous beliefs of varying degrees. Once beliefs are ambiguous, the degree of ambiguity not only can impact choice, but the degree to which the decision maker is pessimistic or ambiguity averse also becomes relevant. Indeed, using our $\alpha$-MMEU model, we show a change in either the degree of ambiguity or the degree of pessimism can lead a decision maker to “convert” from one religion to another. We illustrate the model’s key and unique insights using an example where the decision maker perceives three religious alternatives.

The article proceeds as follows. In Sect. 2, we motivate the application of decision theory to religious choice, and motivate moving beyond SEU theory to allow for ambiguity. The model is presented in Sect. 3. The interaction between the degree of ambiguity and the degree of pessimism and their effects on religious choice are presented in Sect 4. We conclude with some discussion.

2 Religion, choice, and ambiguity

Applying decision theory implicitly assumes people choose their religion, yet many would reject this notion. Within Christianity, for example, different sects have arisen because of different answers to the question, “Do you choose God, or does God choose you?” In the 1552 “Consent” concerning the “Eternal Predestination of God”…, protestant reformer John Calvin writes, “…it ought to have entered the minds…from whence faith comes;…it flows from Divine election as its eternal source” Calvin (1552 [2007]). This Calvinist perspective, which is roughly equivalent to the perspective of the Catholic theologian Augustine, is that God chooses you. The prototypical Christian alternative to Calvinism is Arminianism, associated with the
Dutch theologian Jacob Hermann, best known by his Latin name Jacob Arminius. The Arminian doctrine is that God’s grace is offered to all, but can be refused. In other words, people must choose God. Beyond Christianity, other religions around the world, and sects within them, vary in their beliefs about the ability one has to choose a faith or religion. In constructing a model that assumes one’s religion is chosen, we intend no disrespect to any faith. Indeed, this assumption may be wrong.

Whether or not we are examining “rational” choice is also open to question because rational choice is typically associated with scientific reasoning. Science cannot effectively help us understand what happens to us when we die, for there is no generally accepted database on after-death experiences. If there is life after death, we do not get much of a glimpse of it while we are alive. We can readily develop theories that explain what happens after death, but we cannot readily reject even a most incredible theory by testing it against facts.

Pascal argued “prudence” can provide insight for religious choice that scientific “reason” cannot (Jorden 1994a). In essence, he argued people should apply decision theory because they cannot apply science. Pascalian prudence is purely logical, deriving a choice without reference to facts, a form of rationality fundamentally different than scientific rationality that is both logical and empirical, testing theory against observations.

While our theory, like Pascal’s, has potential to explain the observed prevalence of particular religions, it is more sensible to view decision theories of religious choice as normative rather than positive. Decision theory may explain religious choice because it is an approach for making a decision on religion. The logic in Pascal’s wager suggests that Theism will be more popular than Atheism, which is indeed the case and always has been. Likewise, our model suggests empirically testable hypotheses about the relative prevalence of different religions and offers predictions about the characteristics of the adherents of particular religions. However, the usual reason for expecting rational choice models to have predictive power is the expectation that a survival of the fittest process will weed out nonrational choices. With religious choice, the lack of feedback from beyond the grave largely precludes the weeding out of false religions. This not only gives reason to doubt the predictive power of any theory of religious choice but also offers an explanation for why there is such religious proliferation.5

The lack of feedback from beyond the grave is particularly problematic with regard to belief formation. To demonstrate the difference between risk and uncertainty,6 Knight (1921[1971]) provides an example of a manufacturer contemplating an increase in capacity and argues: “What is the “probability” of error (strictly, of any assigned degree of error) in the judgment? It is manifestly meaningless to speak of either calculating such a probability a priori or of determining it empirically by studying a large number of instances. The essential and outstanding fact is that the “instance” in question is so entirely unique that there are no others or not a sufficient number to make it possible to tabulate enough like it to form a basis for any inference of value about

5 We thank an anonymous reviewer for helping us recognize that our theory is primarily normative and not likely to meet with the predictive success of typical rational choice theories because of the relatively unique nature of religious choice.

6 The distinction between risk and uncertainty was also made by Keynes (1921).

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any real probability in the case we are interested in.” Religious choice has the uniqueness quality of which Knight speaks, suggesting decision makers will have difficulty inferring probabilities regarding validity of different religions.

A contribution of Savage (1954) was to show that, if a particular set of axioms is satisfied, then a decision maker’s beliefs regarding the uncertainty can be represented by a unique personal or subjective probability distribution. However, in the spirit of Knight, Montgomery (1996) questions the applicability of SEU because one cannot have confidence that a reasonable set of beliefs can be formed. He notes that it has become standard in economics to place restrictions on individual belief formation. In particular, the “common prior assumption” is typically employed so any differences in beliefs arise from the effect of asymmetric information on belief updating. Montgomery argues the usual Bayesian updating suffers in the context of religion because we are not only dealing with the unknown but the unknowable. As Shakespeare (Hamlet, Act 3) noted, death is “the undiscovered country from whose bourn no traveler returns.” Montgomery suggests abandoning SEU theory in favor of cognitive dissonance theory for explaining religious choice.

An alternative is to abandon updating and represent ambiguity by with a set of probability measures. Gilboa et al. (2007) note that the decision maker will often not have information that allows an accurate formation of the initial prior, common or otherwise. It was this observation that led Gilboa and Schmeidler (1989) to offer a model in which the decision maker entertains a set of priors, rather than a single one, which is the standard way ambiguity is now modeled. Religious choice is inherently ambiguous because it is inherent that such needed information is lacking.

“Coarse contingencies” as described by Epstein et al. (2007) are another potential source of ambiguity. They state, “The impossibility of fully describing all relevant contingencies is one reason, an important one in our view, why decision makers may not be able to quantify uncertainty about future payoffs with a single probability measure.” Gilboa et al. (2007) similarly critic the standard SEU approach, noting “economics is the only discipline…which…adopts…the…tenet…[that] the state space is elaborate enough to be able to describe anything of relevance.” Epstein et al. (2007) axiomatize a counterpart of Gilboa and Schmeidler (1989) multiple-priors utility without relying on exogenously specified state space of SEU theory. Our use of the $\alpha$-MMEU model, which captures ambiguity in the form of multiple priors, can be rationalized because we would expect many decision makers would make religious choices “without having in mind a complete list of all the contingencies that could influence outcomes.”

Even if all contingencies are recognized, ambiguity may arise because the decision maker is not confident about the SEU probability measure obtained. Savage himself recognized that “…there seem to be some probability relations about which we feel relatively “sure” as compared with others…The notion of “sure” and “unsure” introduced here is vague, and my complaint is precisely that neither the theory of personal probability, as it is developed in this book, nor any other device known to me renders the notion less vague” (Savage 1954, pp. 57–58). He then points out that “One tempting representation of the unsure is to replace the person’s single probability measure $P$ by a set of such measures…” (Savage 1954, p. 58). Because the $\alpha$-MMEU model allows for a set of probability measures, rather than only a singleton measure, it can capture ambiguity associated with the lack of confidence a decision maker may possess.
Models of choice under ambiguity have largely been developed to explain Ellsberg (1961)-type behavior, contradictions to SEU theory that have repeatedly and routinely been found in the experimental and empirical literature.\(^7\) If a decision maker includes all contingencies yet perceives ambiguity, at least one of Savage’s axioms does not hold.\(^8\) The potential for such an axiom violation provides a final motivation for the existence of ambiguity in religious choice.

To illustrate, consider Ellsberg’s (1961) two-color example, and associate choosing the correct ball color with choosing the true religion. In general, human subjects would rather choose from the “risky urn” containing 50 white and 50 black balls than from the “ambiguous urn” containing 100 balls where the mixture of black and white is unknown. This preference indicates that the typical decision maker responds to ambiguity by reducing probability associated with the event, regardless of the event. Summing across all events, this implies the probabilities are nonadditive (i.e., do not sum to one), a violation of Savage’s Sure Thing principle and probabilistic sophistication.

We now proceed to construct a model where religious choice can be examined with the degree of ambiguity varying from no ambiguity to complete ambiguity.

### 3 Parameterizing ambiguity in an \(\alpha\)-MMEU model

Assume a decision maker, DM, must choose an action \(x\) from \(N\) mutually exclusive actions \(X = \{1, 2, \ldots, N\}\). DM perceives the payoff consequence provided by action \(x\) depends on which event \(\theta\) arises from among \(N\) mutually exclusive events \(\Theta = \{1, 2, \ldots, N\}\). In particular, DM perceives the payoff \(u^x_{\theta}\) will be received when action \(x\) is chosen and event \(\theta\) occurs.

To apply this framework as a model of religious choice, let the action be the choice to adopt religion \(x \in X\). DM is able to arrange his or her conceptions of different religions into \(N\) mutually exclusive alternatives. DM assumes one and only one of these is the true religion. Uncertain as to which is true, DM perceives \(N\) possible events, where \(\theta \in \Theta\) is the event “religion \(\theta\) is true.” DM perceives the payoff \(u^x_{\theta}\) is the consequence of adopting religion \(x\) under event \(\theta\). To have a specific example, suppose DM perceives three religious alternatives with the payoffs presented in Table 1.

DM’s beliefs may be ambiguous, meaning the likelihoods of the different events in \(\Theta\) may not be known with precision. These beliefs are represented by the set of probability distributions \(P\), where \(P\) is a subset of the \(N\) dimensional probability simplex \(\Delta \equiv [0, 1]^N\). A generic element of \(P\) is a probability distribution \(p = (p_1, p_2, \ldots, p_N)\), and one can think of \(P\) as representing both information and confidence in that information.\(^9\) If DM has no information that allows a winnowing of the

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\(^7\) Camerer and Weber (1992) and Camerer (1999) provide accessible introductions to the empirical literature on the Ellsberg Paradox.

\(^8\) We thank an anonymous reviewer for this insight, and for the encouragement to present more specifically why religious choice is not only risky, but ambiguous.

\(^9\) Gajdos et al. (2004) provide a complementary interpretation of the set of probabilities. The decision maker in their model maximizes the minimum expected utility computed with respect to a subset of the set of initially given priors. The extent to which the set of initially given priors is reduced is a measure of aversion to information imprecision.
set $P$, then there is complete ambiguity, characterized by $P = \Delta$. DM only knows the probability of each state is between 0 and 1. At the other extreme, DM has information and confidence in it which leads to the belief that only one distribution is valid. The set $P$ is a singleton, and DM faces pure risk.

DM has alpha-maximin expected-utility preferences, ¹⁰ abbreviated as $\alpha$-MMEU. This implies the utility $V(x)$ from action $x$ is given by

$$V(x) = \alpha \min_{p \in P} \left\{ \sum_{\theta \in \Theta} p_\theta u_\theta^x \right\} + (1 - \alpha) \max_{p \in P} \left\{ \sum_{\theta \in \Theta} p_\theta u_\theta^x \right\}.$$ (1)

The $\alpha$-MMEU model (see, e.g., Cohen 1992; Jaffray and Philippe 1997; Ghirardato et al. 1998; Ghirardato et al. 2004; Siniscalchi 2006; Ludwig and Zimper 2006; Olszewski 2007; Eichberger et al. 2008) is a generalization of the MMEU model. The general nature of $\alpha$-MMEU model makes it especially suitable for modeling religious choice. Beliefs are represented by a set of probability distributions. The model reduces to the Savage (1954) SEU model when the set of distributions is a singleton, capturing a decision maker who happens to be able to construct a probability distribution over different religious states. At the opposite extreme, it reduces to the Arrow and Hurwicz (1972) model of complete ambiguity when the set of distributions becomes the probability simplex, capturing a decision maker who regards religious choice as being completely ambiguous. Between these two extremes, the size of the set of probability distributions is a measure of the degree of ambiguity, or a measure of what Gajdos et al. (2004) refer to as the degree of information imprecision.

When $\alpha$-MMEU is applied, the expected utility associated with a given action $x$ is calculated for each admissible probability distribution $p \in P$, with all resulting calculations being discarded except for the minimum and maximum. The parameter $\alpha = [0, 1]$ characterizes DM’s degree of pessimism, as in the Arrow–Hurwicz criterion. This parameter determines the weight given to the minimum and maximum expected utilities provided by action $x$. Finding utility $V(x)$ for each action $x$ in this manner, DM selects action $x$ that maximizes $V(x)$.

When $\alpha = 1$, $\alpha$-MMEU preferences have the MMEU form. DM is totally pessimistic. One can say DM exhibits total ambiguity aversion in this case because the existence of multiple potentially applicable probability distributions is translated into the presumption that the least favorable distribution applies. In contrast, $\alpha = 0$ implies DM is totally ambiguity tolerant, presuming the most optimistic probability distribution applies. As $\alpha$ increases from 0 to 1, DM changes from being ambiguity tolerant to being ambiguity intolerant. Although $\alpha$-MMEU is more general than the

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¹⁰ Ghirardato et al. (2004) provide an axiomatic characterization of $\alpha$-MMEU preferences.
MMEU model, it shares some of its shortcomings. Specifically, $\alpha$-MMEU still ignores almost all of the information contained in the DM’s set of priors; preferences are completely characterized by the best and worst scenarios but nothing between these two possibilities.

Imposing the requirement that $P$ is the core of a simple capacity allows a simple parameterization of the degree of ambiguity and, as will become clear below, places our model on a solid axiomatic ground. (See Appendix A for definitions of a capacity, a core of a capacity, and a supermodular capacity.) Given $f \equiv (f(\{1\}), \ldots, f(\{N\})) \in \Delta$ and $\lambda \in [0, 1]$, a capacity $v$ is simple if $v(\{\theta\}) = (1 - \lambda) f(\{\theta\})$ for all $\theta \in \Theta$, $v(\Theta) = 1$, and $v(\{i_1, \ldots, i_k,\}) = \sum_{j=1}^{k} v(\{i_j\})$ when $k < N$. The probability distribution $f$ is called an anchor. A simple capacity is supermodular and $P = \{(1 - \lambda) f\} + \lambda \Delta$. That is, $P$ is the Minkowski sum$^{11}$ of a singleton set $\{(1 - \lambda) f\}$ and the probability simplex $\Delta$ scaled by $\lambda$. These assumptions imply $P$ is a simplex with faces parallel to those of simplex $\Delta$, and the variable $\lambda$ is naturally interpreted as a measure of the degree of ambiguity. In Bayesian statistics, set $P$ is called an $\varepsilon$-contaminated set of priors (Berger and Berliner 1986).

Simple capacities constitute a special case of neo-additive capacities (Chateauneuf et al. 2007). Recently, Chateauneuf et al. (2007) have axiomatized CEU preferences with neo-additive capacity. In addition to being tractable, a neo-additive capacity “maintains the key features of the weighting schemes consistently observed in experiments” (p. 539). It turns out that DM has CEU preferences with simple capacity $v$ if and only if DM has $\alpha$-MMEU with beliefs given by the core of capacity $v$ (Ludwig and Zimper 2006; Chateauneuf et al. 2007).

Thus, the preferences employed in this article stem from the behavioral axioms employed in Chateauneuf et al. (2007). Under these axioms, the coefficient of pessimism $\alpha$ is constant over the space of all acts.

This is in contrast to Ghirardato et al. (2004) axiomatization of $\alpha$-MMEU preferences where if one takes the set of probability distributions derived from their partial order of independent acts as DM’s beliefs then coefficient $\alpha$ in general depends on the act over which the $\alpha$-MMEU preference functional is evaluated (Eichberger et al. 2008). Since having a constant $\alpha$ considerably simplifies our comparative static analysis, we have adopted belief structure represented by a simple capacity.

Moreover, as stated in the following theorem (for a proof, see Ludwig and Zimper 2006), the $\alpha$-MMEU preference functional with respect to the core of a simple capacity has a compact and easy-to-analyze form.

**Theorem 1** (Parameterizing ambiguity) Let $\theta_i$ be defined by $\theta_i \in \{1, \ldots, N\}$, $\theta_i \neq \theta_j$ for $i \neq j$ with $i, j \in \{1, \ldots, N\}$, and $u_{\theta_1}^{x} \leq u_{\theta_2}^{x} \leq \cdots \leq u_{\theta_{N-1}}^{x} \leq u_{\theta_N}^{x}$. If $P$ is the core of simple capacity $v$, then

$$
V(x) = (1 - \lambda) \left\{ f (\{\theta_1\}) u_{\theta_1}^{x} + f (\{\theta_2\}) u_{\theta_2}^{x} + \cdots + f (\{\theta_N\}) u_{\theta_N}^{x} \right\} 
+ \lambda \left\{ u_{\theta_N}^{x} - \alpha \left( u_{\theta_N}^{x} - u_{\theta_1}^{x} \right) \right\}. 
$$

$^{11}$ The Minkowski sum of sets $A$ and $B$ is defined as $A + B = \{x + y : x \in A, y \in B\}$.

$^{12}$ Note that we slightly abuse our notation by occasionally subsuming the dependence of $\theta$ on the choice of action $x$. 

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Theorem 1 indicates that, when the set of beliefs \( P \) is restricted to the core of a simple capacity, the \( \alpha \)-MMEU utility level for an action is a convex combination of the utility obtained from applying the anchor probability distribution and the utility obtained when all probability distributions are admissible. Because the parameter \( \lambda \) is the weight given to these two utility extremes, Theorem 1 indicates it is a measure of the degree of ambiguity when preferences are \( \alpha \)-MMEU.

To illustrate the applicability of Theorem 1 to our model of religious choice, consider three special cases: (1) \( \lambda = 0 \); (2) \( \lambda = 1, \alpha = 1 \); and (3) \( \lambda = 1, \alpha = 0 \). For case 1, there is no ambiguity, the anchor probability distribution applies, the degree of pessimism does not affect the evaluation of the alternatives, and the decision criterion reduces to subjective expected utility. Let the anchor probability distribution be \( f = (0.8, 0.1, 0.1) \) for religions \( X = (a, b, c) \) presented in Table 1. Then, utility representation (2) implies \( V(a) = 4, V(b) = -7, \) and \( V(c) = -14, \) and religion \( a \) is chosen. For case 2, ambiguity is complete and DM resolves uncertainty using max–min decision rule because he/she is a total pessimist. The utility function (2) implies \( V(a) = -20, V(b) = -10, \) and \( V(c) = -20, \) and religion \( b \) is chosen. For case 3, ambiguity is complete and DM resolves uncertainty using max–max decision rule because he/she is a total optimist. The utility function (2) implies \( V(a) = 10, V(b) = 20, \) and \( V(c) = 30, \) and religion \( c \) is chosen. These extreme examples illustrate how changes in the degree of ambiguity and changes in the degree of pessimism can alter religious choice. Next, we delineate how these two factors interact.

4 Interacting effects of the degree of pessimism and the degree of ambiguity on utility

Having a continuous measure of the degree of ambiguity allows us to examine its marginal effect on utility. More interestingly, we can examine how a change in the degree of ambiguity interacts with a change in the degree of pessimism. The marginal effect of the degree of ambiguity on the utility provided by a particular action \( x \) is given by

\[
\frac{\partial V(x)}{\partial \lambda} = - \left\{ f (\{ \theta_1 \}) u^\lambda_{\theta_1} + f (\{ \theta_2 \}) u^\lambda_{\theta_2} + \cdots + f (\{ \theta_N \}) u^\lambda_{\theta_N} \right\} + \left\{ \alpha u^\lambda_{\theta_1} + (1 - \alpha) u^\lambda_{\theta_N} \right\}. \tag{3}
\]

Using (3), the following result is readily obtained. (See Appendix A for a proof.).

Theorem 2 (Ambiguity, pessimism, and utility) For any set of beliefs \( P \) and any action \( x \in X \), there exists a degree of pessimism \( \hat{\alpha}(x) \) such that \( \alpha \left( \begin{array}{c} \geq \\ = \\ \leq \end{array} \right) \hat{\alpha}(x) \iff \frac{\partial V(x)}{\partial \lambda} \left( \begin{array}{c} \geq \\ = \\ \leq \end{array} \right) 0, \text{ and} \)
Theorem 2 indicates that the qualitative impact of a change in the degree of ambiguity depends on the degree of pessimism. To examine the interesting cases, assume there is some variation in the payoffs so \( u_{\theta_N}^x > u_{\theta_1}^x \) and the strict inequality of the theorem holds. When DM is sufficiently pessimistic, the utility of any action decreases as ambiguity increases. Conversely, when DM is sufficiently optimistic, the utility of any action increases as the degree of ambiguity increases. At the threshold degree of pessimism \( \hat{\alpha} \), a change in the degree of ambiguity does not affect the utility associated with action \( x \). This threshold is related to the payoffs and the anchor probability distribution by

\[
\hat{\alpha}(x) \equiv \frac{u_{\theta_N}^x - \left\{ f(\theta_1)u_{\theta_1}^x + f(\theta_2)u_{\theta_2}^x + \ldots + f(\theta_N)u_{\theta_N}^x \right\}}{u_{\theta_N}^x - u_{\theta_1}^x}.
\]

The implied \((\alpha, \lambda)\) isopayoff mapping is presented in Fig. 1, and the calculations that prove the mapping has this structure are presented in Appendix B.

Theorem 2 is an intuitive result. An increase in ambiguity enlarges the set of admissible probability distributions \( P \). Some of the added distributions will assign higher probabilities to outcomes that are more attractive and some will assign higher probabilities to outcomes that are less attractive. This implies \( \max_{p \in P} \left\{ \sum_{\theta \in \Theta} p_{\theta} u_{\theta}^x \right\} \) in the preference functional (1) will increase, while \( \min_{p \in P} \left\{ \sum_{\theta \in \Theta} p_{\theta} u_{\theta}^x \right\} \) will decrease. A more optimistic decision maker places more weight on the former, whereas the pessimistic decision maker places more weight on the latter. Thus, an increase in ambiguity will increase utility when the optimist is optimistic enough, whereas it will decrease utility when the pessimist is pessimistic enough.

**Fig. 1** Isopayoff mapping for the degree of pessimism \( \alpha \) and degree of ambiguity \( \lambda \).
To apply Theorem 2 to our theory of religious choice, consider religion \(a\) in Table 1. The extreme payoffs are \(u^a_{\theta_1} = 10\) and \(u^a_{\theta_2} = -20\), and the expected utility of religion \(a\) under no ambiguity is \(f(\{\theta_1\}) u^a_{\theta_1} + f(\{\theta_2\}) u^a_{\theta_2} + \cdots + f(\{\theta_N\}) u^a_{\theta_N} = (0.8)(10) + (0.1)(-20) + (0.1)(-20) = 4\). This implies \(\hat{\alpha}(a) = [10 - 4] / [10 - (-20)] = 0.20\). For religions \(b\) and \(c\), \(\hat{\alpha}(b) = 0.90\) and \(\hat{\alpha}(c) = 0.88\). Thus, Theorem 2 indicates DM must be quite optimistic (\(\alpha < 0.20\)) for an increase in ambiguity to increase the utility associated with religion \(a\), while DM can be rather pessimistic (e.g., \(\alpha = 0.87\)) and an increase in ambiguity will increase the utilities associated with religions \(b\) and \(c\). With a moderate degree of pessimism (e.g., \(0.21 < \alpha < 0.87\)), an increase in ambiguity will decrease the utility associated with religion \(a\) while increasing the utilities associated with religions \(b\) and \(c\). This illustrates the finding that, because the marginal impact of a change in the degree of ambiguity varies across the religions, a change in ambiguity can change DM’s religious choice.

5 The anchor choice, max–min choice, and max–max choice

In the \(\alpha\)-MMEU model, the anchor choice, the max–min choice, and the max–max choice are particularly important reference point choices. In this section, we delineate how the degree of ambiguity and the degree of pessimism determine the \(\alpha\)-MMEU decision maker’s choice relative to these three reference choices.

For analysis convenience, we rewrite (2) in the form

\[
V(x; \lambda, \alpha) = (1 - \lambda) A(x) + \lambda B(x, \alpha),
\]

(4)

where \(A(x) \equiv f(\{\theta_1(x)\}) u^x_{\theta_1(x)} + f(\{\theta_2(x)\}) u^x_{\theta_2(x)} + \cdots + f(\{\theta_N(x)\}) u^x_{\theta_N(x)}\) and \(B(x, \alpha) \equiv u^x_{\theta_N(x)} - \alpha \left(u^x_{\theta_N(x)} - u^x_{\theta_1(x)}\right)\). Now, fixing the degree of pessimism at an arbitrary level \(\alpha \in [0, 1]\), consider two actions \(x\) and \(y\). If \(A(x) \geq A(y)\) and \(B(x, \alpha) \geq B(y, \alpha)\), then action \(x\) is preferred to \(y\) irrespective of the degree of ambiguity. If \(A(x) > A(y)\) and \(B(x, \alpha) < B(y, \alpha)\), then action \(x\) is preferred to \(y\) if and only if DM has sufficiently unambiguous beliefs. In general, DM will choose the action for which \(A\) is highest when beliefs are sufficiently unambiguous, while the action for which \(B\) is highest will be chosen when beliefs are sufficiently ambiguous. When the degree of ambiguity is high, the degree of pessimism points DM either toward the optimistic choice \(\arg \max_x \left\{u^x_{\theta_N(x)}\right\}\) or toward the pessimistic choice \(\arg \max_x \left\{u^x_{\theta_1(x)}\right\}\). These observations are summarized more carefully in the following theorem. (See Appendix A for a proof)

Theorem 3 (Anchor, max–max, max–min) Given any set of beliefs \(P\),

(i) For any degree of pessimism \(\alpha \in [0, 1]\), there exists a degree of ambiguity \(\lambda(\alpha)\) such that DM prefers action \(\arg \max_x A(x)\) whenever \(\lambda < \lambda(\alpha)\);

(ii) For any degree of pessimism \(\alpha \in [0, 1]\), there exists a degree of ambiguity \(\hat{\lambda}(\alpha)\) such that DM prefers action \(\arg \max_x B(x, \alpha)\) whenever \(\lambda > \hat{\lambda}(\alpha)\);
(iii) There exist $\alpha_L$ and $\lambda_L$ such that DM prefers action $\arg \max_x \{ u^x_{\theta_N(x)} \}$ whenever $\alpha \leq \alpha_L$ and $\lambda > \lambda_L$; and

(iv) There exist $\alpha_H$ and $\lambda_H$ such that DM prefers action $\arg \max_x \{ u^x_{\theta_1(x)} \}$ whenever $\alpha \geq \alpha_H$ and $\lambda > \lambda_H$.

Theorem 3 indicates pessimistic (optimistic) decision makers are biased toward the max–min (max–max) choice, and increasingly so as ambiguity increases. When there is complete ambiguity ($\lambda = 1$), there is a threshold level of pessimism that will lead DM to choose the pessimistic max–min action $\arg \max_x \{ u^x_{\theta_1(x)} \}$, and this choice is also best if DM becomes increasingly pessimistic. When the decision maker is entirely pessimistic, the max–min action is best under complete ambiguity and under less than complete ambiguity down to some threshold level $\lambda_L$. By analogous reasoning, there are thresholds $\alpha_H$ and $\lambda_H$ that are associated with DM choosing the optimistic max–max action $\arg \max_x \{ u^x_{\theta_N(x)} \}$.

Figure 2 pictorially illustrates the implications of Theorem 3. The tradeoffs between the level of pessimism and level of ambiguity derived in Theorem 2 indicate there are sets of $(\lambda, \alpha)$ combinations that support the max–min and max–max choices, as shown. As the level of ambiguity decreases to zero, Theorem 3 indicates there must be a set of $(\lambda, \alpha)$ combinations that support the anchor choice $\arg \max A(x)$, as shown. Figure 2 is drawn under the assumption that the decision maker perceives three religious alternatives, and under the assumption that one of these three is the anchor choice, the second is the max–min choice, while the third is the max–max choice. The boundaries of the three regions in Fig. 2 will vary with the payoff structure perceived by the decision maker.

To further illustrate Theorem 3, reconsider the religious payoff parameter values in Table 1. When $\lambda = 0$, $V(a; 0, \alpha) = A(a) = 4$, $V(b; 0, \alpha) = A(b) = -7$, and
\( V(c; 0, \alpha) = A(c) = -14 \). In this case, the optimal choice is the anchor choice, religion \( a \). In this no ambiguity case, the degree of pessimism does not affect the religious choice. However, consider now a low degree of pessimism (\( \alpha = .1 \)) and increase the degree of ambiguity. When we hit the ambiguity level \( \lambda = .51 \), we find \( V(a; .51, .1) = 5.53 \), \( V(b; .51, .1) = 5.24 \), and \( V(c; .51, .1) = 5.89 \), so the optimal choice switches from religion \( a \) to the max–min choice, religion \( c \). The optimal choice remains religion \( c \) for the degree of ambiguity in the range \( \lambda = (.50, 1] \). When the degree of pessimism is high (\( \alpha = .9 \)) and we increase the degree of ambiguity, when we hit the ambiguity level \( \lambda = .53 \), we find \( V(a; .53, .9) = -7.13 \), \( V(b; .53, .9) = -7.00 \), and \( V(c; .53, .9) = -14.53 \), so that the optimal choice switches from religion \( a \) to the max–min choice, religion \( b \). The optimal choice remains religion \( b \) for the degree of ambiguity in the range \( \lambda = (.52, 1] \).

Under complete ambiguity, the preference functional (4) becomes \( V(x; 1, \alpha) = B(x, \alpha) \). We can write \( B(x, \alpha) = \alpha u^x_{\theta_1(x)} + (1 - \alpha) u^y_{\theta_2(y)} \) and \( B(y, \alpha) = \alpha u^y_{\theta_1(y)} + (1 - \alpha) u^x_{\theta_2(x)} \) so \( B(x, \alpha) - B(y, \alpha) = \alpha \left[ u^x_{\theta_1(x)} - u^y_{\theta_2(y)} \right] + (1 - \alpha) \left[ u^y_{\theta_1(y)} - u^x_{\theta_2(x)} \right] \).

It follows that, if \( u^x_{\theta_1(x)} > u^y_{\theta_2(y)} \) and \( u^y_{\theta_1(y)} > u^x_{\theta_2(x)} \), then \( B(x, \alpha) > B(y, \alpha) \) for all \( \alpha \in [0, 1] \). Action \( x \) provides a higher payoff than action \( y \) in the most optimistic state and in the most pessimistic state. Thus, action \( x \) will not be chosen over action \( y \) under complete ambiguity, no matter what the degree of pessimism. This observation suggests the following definition.

**Definition 1** (*Irrelevance under complete ambiguity*) An action \( y \) is “irrelevant under complete ambiguity” if there is some other action \( x \) such that \( u^x_{\theta_N(x)} > u^y_{\theta_N(y)} \) and \( u^y_{\theta_1(y)} > u^x_{\theta_1(x)} \).

Irrelevance is important because it reduces the number of meaningful alternatives when it applies. In particular, for religious choice, the irrelevance of many conceivable religions offers an explanation for why a given decision maker may not perceive too many relevant religions in a decision matrix. It is possible that one particular action makes all other actions irrelevant under complete ambiguity. In this special case, the move toward complete ambiguity motivates DM to select one particular choice, no matter what DM’s degree of pessimism. This special case is described by the following definition.

**Definition 2** (*Dominance under complete ambiguity*) An action \( x \) is “dominant under complete ambiguity” if \( u^x_{\theta_1(x)} > u^y_{\theta_N(y)} \) and \( u^y_{\theta_1(y)} > u^x_{\theta_N(x)} \) for all \( y \in X \), \( y \neq x \).

The concepts of irrelevance and dominance under complete ambiguity have implications for evangelism. If the beliefs of decision makers are completely ambiguous, then one religion will make others irrelevant by instilling the perception that the religion offers a high reward for adoption when it is true, and high penalty for non-adoption. For example, suppose the decision matrix for DM changes from what is presented in Table 1 to what is presented in Table 2. As before, religion \( c \) offers the highest reward for adoption when it is true: \( u^c_{\theta_N(c)} = 30 > u^b_{\theta_N(b)} = 20 > u^a_{\theta_N(a)} = 10 \). However, now religion \( c \) also imposes the largest penalty for nonadoption when true: \( u^c_{\theta_1(c)} = -20 > u^b_{\theta_1(b)} = -30 = u^c_{\theta_1(c)} = -30 \). It follows that religion \( c \) is dominant under complete ambiguity, making both religions \( a \) and \( b \) irrelevant.
Table 2 Dominance under complete ambiguity

<table>
<thead>
<tr>
<th></th>
<th>Religion a</th>
<th>Religion b</th>
<th>Religion c</th>
</tr>
</thead>
<tbody>
<tr>
<td>Religion a</td>
<td>10</td>
<td>-20</td>
<td>-30</td>
</tr>
<tr>
<td>Religion b</td>
<td>-10</td>
<td>20</td>
<td>-30</td>
</tr>
<tr>
<td>Religion c</td>
<td>-10</td>
<td>-20</td>
<td>30</td>
</tr>
</tbody>
</table>

When $\lambda < 1$, DM’s choice is affected by the difference between the payoffs for the most preferred and the least preferred states of nature, $u_{\theta_N}^x - u_{\theta_1}^x$. This is because the marginal effect of the degree of pessimism on the attractiveness of action $x$ is $-\lambda \left(u_{\theta_N}^x - u_{\theta_1}^x\right) < 0$, obtained by differentiating (2) with respect to $\alpha$. Greater pessimism reduces the anticipated payoff associated with any action $x$, as would be expected. More interestingly, we see this marginal impact depends on both the degree of ambiguity $\lambda$ and the payoff difference $u_{\theta_N}^x - u_{\theta_1}^x$. For a given level of ambiguity $\lambda$, greater pessimism reduces utility more when the payoff difference $u_{\theta_N}^x - u_{\theta_1}^x$ is larger. And, for a given payoff difference $u_{\theta_N}^x - u_{\theta_1}^x$, greater pessimism reduces utility more when the degree of ambiguity $\lambda$ is greater. This reasoning underlies the following result, which is proved in Appendix A.

**Theorem 4** (Smallest and largest differential payoffs and the degree of pessimism) If $u_{\theta_N}^x - u_{\theta_1}^x \leq u_{\theta_N}^{x'} - u_{\theta_1}^{x'}$ for all actions $x'$ that are alternatives to action $x$, and DM prefers action $x$ to all alternative actions $x'$ with degree of pessimism $\tilde{\alpha}$ and beliefs $P$, then DM must prefer action $x$ to all alternative actions $x'$ with higher degree of pessimism $\alpha > \tilde{\alpha}$ and beliefs $P$. Conversely, if $u_{\theta_N}^x - u_{\theta_1}^x \geq u_{\theta_N}^{x'} - u_{\theta_1}^{x'}$ for all actions $x'$ that are alternatives to action $x$, and DM prefers action $x$ to all alternative actions $x'$ with degree of pessimism $\tilde{\alpha}$ and beliefs $P$, then DM must prefer action $x$ to all alternative actions $x'$ with degree of pessimism $\alpha < \tilde{\alpha}$ and beliefs $P$.

Theorem 4 indicates that optimists are more likely than pessimists to find appeal in a choice with a large difference between the largest possible payoff and the lowest. If such a choice is appealing, it will tend to capture the most optimistic decision makers. Conversely, pessimists will tend to find more appeal in a choice where this difference is small. If a choice with such a small difference in the possible payoffs is attractive, Theorem 4 indicates it will be the most pessimistic decision makers who will find it attractive.

To illustrate Theorem 4, when the level of the ambiguity for the decision maker with the Table 1 payoff structure is $\lambda = .50$, we find $V(a; .50, .11) = 5.35$, $V(b; .50, .11) = 4.85$, and $V(c; .50, .11) = 5.25$, so the optimal choice is religion $a$. Religion $a$ has the smallest payoff difference, with $(u_{\theta_N}^a - u_{\theta_1}^a) = 10 - (-20) = 30$. Theorem 4 indicates that religion $a$ would be the optimal choice for any higher degree of pessimism $\alpha \in (.11, 1]$, which trial calculations can confirm. When the level of the ambiguity is $\lambda = .51$, we find $V(a; .51, .1) = 5.53$, $V(b; .51, .1) = 5.24$, and $V(c; .51, .1) = 5.89$, so the optimal choice is religion $c$. Religion $c$ has the largest payoff difference, with $(u_{\theta_N}^c - u_{\theta_1}^c) = 30 - (-20) = 50$. Theorem 4 indicates that religion $c$ would be the optimal choice for any lower degree of pessimism $\alpha \in [0, .10)$, which trial calculations can confirm.
6 Discussion

Using a version of the $\alpha$-MMEU model, we have developed a theory of religious choice under uncertainty. In doing so, we have addressed a concern of Iannaccone (1998, p. 1491) who notes that “the problem of religious uncertainty has received little attention and scarcely any formal analysis.” Our model allows for ambiguity, not just risk, which is significant in that the decision maker’s degree of pessimism has no impact on choice when there is no ambiguity. We have shown that, if religious choice is made under ambiguity, then a change in either the degree of pessimism or the degree of ambiguity can alter the religion choice, while holding constant underlying beliefs and expected payoff consequences.

While people may come to their religious convictions in a variety of ways, applying Pascalian logic is one way. Interestingly, one anecdotal illustration appears in the life of John von Neuman. Unfortunately, in the summer of 1955, he was diagnosed with advanced, incurable cancer. By the time the disease had confined him to bed, von Neumann had converted to Catholicism. About this conversion, Jorden (1994a, p. 1) comments, “As might be expected of the inventor of the minimax principle, von Neumann was reported to have said, perhaps in part jovially, that Pascal had a point: If there is a chance that God exists, and that damnation is the lot of the unbeliever, then it is reasonable to believe.”

While it is clear that we should be careful in expecting our model to have much explanatory power, it is worth asking whether the theory it contains provides a way of organizing observable religious choices. A recent Wikipedia entry reports that Christianity and Islam are the two largest religions in the world, with 2.1 billion and 1.5 billion followers, respectively (Wikipedia 2008). By comparison, the number of Atheists is small, included among the 1.1 billion classified in the category “atheist/anti-theistic/anti-religious/secular/agnostic.” How might our theory explain these numbers?

Consider a decision maker in our model who perceives Christianity, Islam, and Atheism as religious alternatives. Simplistically speaking, Christianity and Islam each offer a heavenly afterlife to adherents, and each claim a hellish afterlife awaits those who do not adhere. It is therefore reasonable to think the typical decision maker would associate the lowest maximum payoff and lowest minimum payoff with Atheism. This implies Atheism will neither be the max–max choice of the extreme optimist nor the max–min choice of the extreme pessimist. Thus, ambiguity will bias both optimists and pessimists away from Atheism, toward Christianity and Islam. That is, the likelihood that religious choice is quite ambiguous for the typical decision maker, combined with the nonextreme payoff structure of Atheism, is a decision theory explanation for the small number of Atheists relative to Christians and Muslims. (Of course, there are also other possible explanations.)

---

13 John von Neumann lived in the 20th century and Blaise Pascal in the 17th, their lives had parallels. Both were great mathematicians, and both made significant contributions to decision theory. Pascal initiated decision theory with his wager, and did path-breaking work on probability. Von Neumann developed the minimax theorem and sparked modern game theory with Oscar Morgenstern with the book *Theory of Games and Economic Behavior* (Morgenstern and von Neumann 1944[2000]). Both died young, Pascal at 39, von Neuman at 54.
Our model indicates ambiguity increases the saliency of religious reward concepts like heaven and religious punishment concepts like hell. As ambiguity increases, the possibility of heaven attracts optimists, whereas the possibility of hell for nonadoption attracts pessimists. To adopt Atheism, which presents no extreme reward or punishment, a move away from complete ambiguity must be made, but such a move is not sufficient. The decision maker must also have an anchor probability distribution that places little probability weight on the truth of either Christianity or Islam, but much weight on the truth of Atheism. Because our model provides some insight as to the characteristics of a person who adopts a particular religion, it may be worthwhile to examine the empirical validity of the model by comparing observed characteristics of particular adherents to the predictions of the model.

A weakness of the current model is that it does not provide much insight as to why Christianity would be adopted over Islam, or vice versa, because it is not clear how these two religions would differ within our model. However, our model could conceivably be extended using the Fox and Tversky (1995) idea that familiarity reduces ambiguity while unfamiliarity breeds ambiguity. The simplifying assumption applied in our model is that the same degree of ambiguity applies to all religions. Allowing the degree of ambiguity to be different for different religions, perhaps as a consequence of differing degrees of familiarity, may allow a decision theory model to make more refined and accurate predictions of religious choice.

We should note that the parameterization of the $\alpha$-MMEU model we have developed here can be applied to more than religious choice. It is especially suited to situations where one might expect an interaction between the degree of ambiguity and degree of pessimism. In a principle-agent problem, for example, if we think of the decision as that being made by an agent, our model suggests the principle may be able to control the behavior of the agent more effectively by increasing the degree of ambiguity in the agent’s mind. This counter-intuitive result arises from the fact that a move toward ambiguity increases the saliency of the highest and lowest possible outcomes associated with a decision alternative, which the principle may also be able to control.

Finally, we note that our theory of religious choice neither help us determine whether God actually exists nor help us determine whether any particular religion is true. If we can choose our religion independently, our theory cannot tell us whether we are choosing a God of our creation, or choosing a God that has created us. Our theory is consistent with the idea that people create God and religion, and suggests such creations will be more popular if when they recognize how people make choices under varying degrees of ambiguity. However, our theory is also consistent with the existence of a God who values faith, and created people who can express it as they make religious choices while facing the ambiguity associated with death.

Acknowledgements We thank two anonymous reviewers and session participants at the IAREP-SABE World Meeting 2008 for helpful comments.

14 We thank an anonymous reviewer for recognizing this possible extension.
Appendix A

Definition of capacity, core of the capacity, and supermodular: Let \( \Theta \) denote a finite state space \( \{1, \ldots, N\} \) and let \( \Sigma \) denote the \( \sigma \)-algebra of all subsets of \( \Theta \); i.e., \( \Sigma = 2^\Theta \). Let \( \Delta \) denote the set of all additive probability measures over \( \Sigma \). A capacity is a nonadditive set function \( v: \Sigma \to [0,1] \) such that (i) \( v(\emptyset) = 0 \), (ii) \( v(\Sigma) = 1 \), and (iii) \( v(X) \leq v(Y) \) for all \( X, Y \subseteq \Sigma \), and \( X \subseteq Y \). The core of the capacity \( v \) is the closed, convex, and bounded set \( C(v) = \{ p \in \Delta : p(X) \geq v(X), \forall X \in \Sigma \} \). \( v \) is by construction polyhedral. The capacity, \( v \), is supermodular (convex) if \( v(X \cup Y) + v(X \cap Y) \geq v(X) + v(Y) \) for all \( X, Y \subseteq \Sigma \). When \( v \) is supermodular, \( C(v) \) is non-empty, \( v \) is balanced, and \( v(X) = \min \{ p(X) : p \in C(v) \} \) for all \( X \in \Sigma \).

Proof of Theorem 2 Since \( u_{\theta_N}^{x} - u_{\theta_1}^{x} \geq 0 \), the derivative \( \frac{\partial V(x)}{\partial \alpha} \) in (3) is nonincreasing in \( \alpha \). When \( \alpha = 0 \), \( \frac{\partial V(x)}{\partial \alpha} = u_{\theta_N}^{x} - \{ f (\{ \theta_1 \}) u_{\theta_1}^{x} + f (\{ \theta_2 \}) u_{\theta_2}^{x} + \cdots + f (\{ \theta_N \}) u_{\theta_N}^{x} \} \) is nonnegative because \( u_{\theta_1}^{x} \leq u_{\theta_2}^{x} \leq \cdots \leq u_{\theta_{N-1}}^{x} \leq u_{\theta_N}^{x} \), and \( \frac{\partial V(x)}{\partial \alpha} \) is strictly positive when \( u_{\theta_N}^{x} > u_{\theta_1}^{x} \). Analogously, when \( \alpha = 1 \), \( \frac{\partial V(x)}{\partial \alpha} = u_{\theta_N}^{x} - \{ f (\{ \theta_1 \}) u_{\theta_1}^{x} + f (\{ \theta_2 \}) u_{\theta_2}^{x} + \cdots + f (\{ \theta_N \}) u_{\theta_N}^{x} \} \) is nonpositive because \( u_{\theta_1}^{x} \leq u_{\theta_2}^{x} \leq \cdots \leq u_{\theta_{N-1}}^{x} \leq u_{\theta_N}^{x} \), and \( \frac{\partial V(x)}{\partial \alpha} \) is strictly negative when \( u_{\theta_N}^{x} > u_{\theta_1}^{x} \). Because \( \frac{\partial V(x)}{\partial \alpha} \) is a continuous and strictly increasing function of \( \alpha \), there must exist a value \( \hat{\alpha}(x) \) such that \( \alpha < \hat{\alpha}(x) \iff \frac{\partial V(x)}{\partial \alpha} < 0 \), and \( \alpha > \hat{\alpha}(x) \iff \frac{\partial V(x)}{\partial \alpha} > 0 \) when \( u_{\theta_N}^{x} > u_{\theta_1}^{x} \).

Proof of Theorem 3 Part (i) follows from the continuity of the payoff function \( V(x; \lambda, \alpha) \) in \( \lambda \) for each action \( x \) and the fact that arg max \( V(x; \lambda = 0, \alpha) = \arg \max_x A(x) \). Part (ii) follows from the continuity of the payoff function \( V(x; \lambda, \alpha) \) in \( \lambda \) for each action \( x \) and the fact that arg max \( V(x; \lambda = 1, \alpha) = \arg \max_x B(x, \alpha) \). Part (iii) follows from the continuity of the payoff function \( V(x; \lambda, \alpha) \) in \( \lambda \) for each action \( x \) and the fact that arg max \( V(x; \lambda = 1, \alpha = 0) = \arg \max_x \{ u_{\theta_N}^{x} \} \). Part (iv) follows from the continuity of the payoff function \( V(x; \lambda, \alpha) \) in \( \lambda \) for each action \( x \) and the fact that arg max \( V(x; \lambda = 1, \alpha = 1) = \arg \max_x \{ u_{\theta_1}^{x} \} \).

Proof of Theorem 4 We need to only prove the first part, for the proof for the second part is analogous. Assume \( u_{\theta_N}^{x} - u_{\theta_1}^{x} \leq u_{\theta_N}^{x'} - u_{\theta_1}^{x'} \) for all \( x' \). Applying Theorem 1, we can write

\[
V(x, \alpha) = (1 - \alpha) \left\{ f (\{ \theta_1 (x) \}) u_{\theta_1}^{x} + f (\{ \theta_2 (x) \}) u_{\theta_2}^{x} + \cdots + f (\{ \theta_N (x) \}) u_{\theta_N}^{x} \right\} + \alpha \left\{ u_{\theta_N}^{x} - \alpha (u_{\theta_N}^{x} - u_{\theta_1}^{x}) \right\}
\]
and
\[
V(x', \alpha) = (1 - \lambda) \left\{ f \left( \{ \theta_1 (x') \} \right) u_{\theta_1}^{x'} + f \left( \{ \theta_2 (x') \} \right) u_{\theta_2}^{x'} + \cdots + f \left( \{ \theta_N (x') \} \right) u_{\theta_N}^{x'} \right\} + \lambda \left\{ u_{\theta_N}^{x'} - \alpha (u_{\theta_N}^{x'} - u_{\theta_1}^{x'}) \right\}
\]

Let \( W(x, x', \alpha) \equiv V(x, \alpha) - V(x', \alpha) \). By our supposition,

(i) \( W(x, x', \alpha) \geq 0 \) for all \( x' \).

Furthermore, the assumption \((u_{\theta_N}^{x'}) - u_{\theta_1}^{x'}) \leq (u_{\theta_N}^{x'} - u_{\theta_1}^{x'}) \) for all \( x' \).

(ii) \( \frac{\partial W(x, x', \alpha)}{\partial \alpha} = -\lambda \left\{ (u_{\theta_N}^{x'}) - u_{\theta_1}^{x'}) - (u_{\theta_N}^{x'} - u_{\theta_1}^{x'}) \right\} \geq 0 \) for all \( \alpha \) and all \( x' \).

It follows from (i) and (ii) that \( W(x, x', \alpha) \geq 0 \) for all \( x' \) and all \( \alpha \) such that \( \alpha \leq \alpha \leq 1 \). That is, any decision maker with degree of pessimism \( \alpha > \alpha \) must also prefer religion \( x \).

\[ \square \]

Appendix B: Isopayoff curve analysis

Define \( \overline{V} \equiv (1 - \lambda) \left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^{x} + f \left( \{ \theta_2 \} \right) u_{\theta_2}^{x} + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^{x} \right\} + \overline{\lambda} \left\{ u_{\theta_N}^{x} - \overline{\alpha} (u_{\theta_N}^{x} - u_{\theta_1}^{x}) \right\} \) where \( 0 \leq \overline{\alpha}, \overline{\lambda} \leq 1 \), and consider the set of pairs \((\alpha, \lambda)\) such that the decision maker’s payoff from action \( x \) yield the same payoff, given by \( \overline{V} = (1 - \lambda) \left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^{x} + f \left( \{ \theta_2 \} \right) u_{\theta_2}^{x} + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^{x} \right\} + \lambda \left\{ u_{\theta_N}^{x} - \alpha (u_{\theta_N}^{x} - u_{\theta_1}^{x}) \right\} \).

Using these two representations of \( \overline{V} \) to solve for \( \lambda \) in terms of \( \alpha \), we obtain:

\[
\lambda (\alpha) = \frac{\overline{\lambda} \left\{ u_{\theta_N}^{x} - \overline{\alpha} (u_{\theta_N}^{x} - u_{\theta_1}^{x}) \right\} - \left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^{x} + f \left( \{ \theta_2 \} \right) u_{\theta_2}^{x} + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^{x} \right\}}{u_{\theta_N}^{x} - \alpha \left( u_{\theta_N}^{x} - u_{\theta_1}^{x} \right)} - \left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^{x} + f \left( \{ \theta_2 \} \right) u_{\theta_2}^{x} + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^{x} \right\}
\]

The first- and second-order derivatives of \( \lambda (\alpha) \) are given by

\[
\frac{d\lambda}{d\alpha} = \frac{\overline{\lambda} \left( u_{\theta_N}^{x} - u_{\theta_1}^{x} \right) \left[ \left( u_{\theta_N}^{x} - \overline{\alpha} (u_{\theta_N}^{x} - u_{\theta_1}^{x}) \right) - \left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^{x} + f \left( \{ \theta_2 \} \right) u_{\theta_2}^{x} + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^{x} \right\} \right]}{\left[ u_{\theta_N}^{x} - \alpha \left( u_{\theta_N}^{x} - u_{\theta_1}^{x} \right) \right]} - \left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^{x} + f \left( \{ \theta_2 \} \right) u_{\theta_2}^{x} + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^{x} \right\}^2,
\]

\[
\frac{d^2\lambda}{d\alpha^2} = \frac{\overline{\lambda} \left( u_{\theta_N}^{x} - u_{\theta_1}^{x} \right)^2 \left[ \left( u_{\theta_N}^{x} - \overline{\alpha} (u_{\theta_N}^{x} - u_{\theta_1}^{x}) \right) - \left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^{x} + f \left( \{ \theta_2 \} \right) u_{\theta_2}^{x} + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^{x} \right\} \right]}{\left[ u_{\theta_N}^{x} - \alpha \left( u_{\theta_N}^{x} - u_{\theta_1}^{x} \right) \right]} - \left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^{x} + f \left( \{ \theta_2 \} \right) u_{\theta_2}^{x} + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^{x} \right\}^3.
\]

The slope and curvature of the isopayoff curve \( \lambda (\alpha) \) depends on the relationship between \( \left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^{x} + f \left( \{ \theta_2 \} \right) u_{\theta_2}^{x} + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^{x} \right\} \) and \( \left\{ u_{\theta_N}^{x} - \overline{\alpha} (u_{\theta_N}^{x} - u_{\theta_1}^{x}) \right\} \).

We have the following three cases:

\[ \square \]
Case 1 \[
\left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^x + f \left( \{ \theta_2 \} \right) u_{\theta_2}^x + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^x \right\} < \left\{ u_{\theta_N}^x - \alpha \left( u_{\theta_N}^x - u_{\theta_1}^x \right) \right\}.
\]

It is straightforward to verify that \( \frac{d\lambda}{d\alpha} > 0 \) and \( \frac{d^2\lambda}{d\alpha^2} > 0 \) over the admissible range of parameters. The isopayoff curve for Case 1 is graphed in Fig. B1.

Case 2 \[
\left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^x + f \left( \{ \theta_2 \} \right) u_{\theta_2}^x + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^x \right\} > \left\{ u_{\theta_N}^x - \alpha \left( u_{\theta_N}^x - u_{\theta_1}^x \right) \right\}
\]
\( \frac{d\lambda}{d\alpha} < 0 \) and \( \frac{d^2\lambda}{d\alpha^2} > 0 \) over the admissible range of parameters (see Fig. B2).

Case 3 \[
\left\{ f \left( \{ \theta_1 \} \right) u_{\theta_1}^x + f \left( \{ \theta_2 \} \right) u_{\theta_2}^x + \cdots + f \left( \{ \theta_N \} \right) u_{\theta_N}^x \right\} = \left\{ u_{\theta_N}^x - \alpha \left( u_{\theta_N}^x - u_{\theta_1}^x \right) \right\}
\]
The isopayoff curve for Case 3 is the vertical line graphed in Fig. B3.

We can graph the three different types of indifference curves for the three possible cases in one figure, which is done in Fig. 1 in the text in the discussion after Theorem 2.

Using isopayoff curves to compare actions \( x \) and \( x' \)

Consider two actions \( x \) and \( x' \) and an arbitrary \((\alpha, \lambda)\). Suppose that action \( x \) yields a higher payoff than action \( x' \) when degree of pessimism and degree of ambiguity are
Fig. B2  The isopayoff curve for case 2

\[ M = \frac{\bar{\lambda} \left[ f(\{\theta_1\})u_{\theta_1}^x + f(\{\theta_2\})u_{\theta_2}^x + \ldots + f(\{\theta_N\})u_{\theta_N}^x \right] - \left\{ u_{\theta_N}^x - \bar{\alpha}(u_{\theta_N}^x - u_{\theta_N}^x) \right\}}{f(\{\theta_1\})u_{\theta_1}^x + f(\{\theta_2\})u_{\theta_2}^x + \ldots + f(\{\theta_N\})u_{\theta_N}^x - \left\{ u_{\theta_N}^x \right\}} \]

\[ R = \bar{\lambda} \alpha + (1 - \bar{\lambda}) \left[ u_{\theta_N}^x \right] - \frac{\left\{ f(\{\theta_1\})u_{\theta_1}^x + f(\{\theta_2\})u_{\theta_2}^x + \ldots + f(\{\theta_N\})u_{\theta_N}^x \right\}}{\left( u_{\theta_N}^x - u_{\theta_N}^x \right)} \]

Fig. B3  The isopayoff curve for case 3

\[-\frac{\alpha}{\alpha} = \hat{\alpha} = \frac{u_{\theta_N}^x - \left\{ f(\{\theta_1\})u_{\theta_1}^x + f(\{\theta_2\})u_{\theta_2}^x + \ldots + f(\{\theta_N\})u_{\theta_N}^x \right\}}{\left( u_{\theta_N}^x - u_{\theta_N}^x \right)}\]
Fig. B4  The set of parameters for which $x$ is preferred to $x'$

given by $(\alpha, \lambda)$. That is, we assume that $(1 - \lambda)A(x) + \lambda B(x, \alpha) \geq (1 - \lambda)A(x') + \lambda B(x', \alpha)$. For the case of parameters depicted in Fig. B4, shaded area represents the set of $(\alpha, \lambda)$ such that $x$ yields a higher payoff than action $x'$.

References


