Introduction to Functions, Sequences, Metric and Topological Spaces, Continuity, Semicontinuity, and Hemicontinuity

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1 Functions

Definition 1 *A* ***function*** f *from (or on) a set* X *to (or into) a set* Y *is a rule that assigns to each* $x \in X$ *a unique element* $f(x)$ *in* Y.

Definition 2 *The collection* G *of pairs* $(x, f(x))$ *in* $X \times Y$ *is called the* ***graph*** *of the function* f.

The word "mapping" is often used for "function".

We will express the fact that f is a function from X into Y by writing

$$f : X \to Y.$$

Definition 3 *The set* X *is called the* ***domain*** *of* f. *The set of values taken by* f, *that is, the set*

$$\{y \in Y : \exists x \text{ such that } y = f(x)\}$$

is called the ***range*** *of* f.

Definition 4 *For* $A \subseteq X$, *the* ***image*** $f(A)$ *of* A *under* f *is the set of elements of* Y *such that* $y = f(x)$ *for some* $x \in A$:

$$f(A) = \{y \in Y : \exists x \in A \text{ such that } y = f(x)\}.$$

Definition 5 *Function* f *is* ***onto*** Y *if and only if* $Y = f(X)$.

Definition 6 *For* $B \subseteq Y$, *the* ***inverse image*** $f^{-1}(B)$ *of* B *is the set of those elements* $x \in X$ *for which* $f(x) \in B$:

$$f^{-1}(B) = \{x \in X : f(x) \in B\}.$$

Definition 7 *Function* f *is called* ***one-to-one*** Y *(or univalent, or injective) if and only if* $f(x_1) = f(x_2)$ *only if* $x_1 = x_2$.

Definition 8 *Functions* $f : X \to Y$ *which are one-to-one and onto are called* ***one-to-one correspondences*** *(or bijective).*

Definition 9 *If function* f *is a one-to-one correspondence between* X *and* Y *then there exists function* $g : Y \to X$ *such that* $g(f(x)) = x$ *and* $f(g(y)) = y$. *The function* g *is called the* ***inverse*** *of* f *and is frequently denoted by* f^{-1}.

Definition 10 *Let* $f : X \to Y$ *and* $g : Y \to Z$. *The function* $h : X \to Z$ *defined as* $h(x) = g(f(x))$ *for all* $x \in X$ *is called the* ***composition*** *of* g *with* f *and is denoted by* $g \circ f$.

Definition 11 *Let* $f : X \to Y$ *and* $A \subseteq X$. *The function* $g : A \to Y$ *defined as* $g(x) = f(x)$ *for all* $x \in A$ *is called the* ***restriction of function*** f *to* A *and is sometimes denoted by* $f|A$.

Exercise Set 1

Exercise 1: Let $f : X \to Y$. Show that f is onto if there is a mapping $g : Y \to X$ such that $f \circ g$ is the identity map on Y, that is, $f(g(y) = y$ for all $y \in Y$.

Exercise 2: Show that:

(i) $f\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right] = \bigcup_{\lambda\in\Lambda} f\left[A_\lambda\right];$

(ii) $f\left[\bigcap_{\lambda\in\Lambda} A_\lambda\right] \subseteq \bigcap_{\lambda\in\Lambda} f\left[A_\lambda\right];$

(iii) Provide an example where $f\left[\bigcap_{\lambda\in\Lambda} A_\lambda\right] \neq \bigcap_{\lambda\in\Lambda} f\left[A_\lambda\right]$

Exercise 3: Show that:

(i) $f^{-1}\left[\bigcup_{\lambda\in\Lambda} A_\lambda\right] = \bigcup_{\lambda\in\Lambda} f^{-1}\left[A_\lambda\right];$

(ii) $f^{-1}\left[\bigcap_{\lambda\in\Lambda} A_\lambda\right] = \bigcap_{\lambda\in\Lambda} f^{-1}\left[A_\lambda\right];$

2 Metric Spaces

Definition 12 *A **metric** on a nonempty set X is a function $d : X \times X \rightarrow \Re_+$ satisfying the following properties:*
a) Non-negativity: For all $x, y \in X$, $d(x, y) \geq 0$.
b) Discrimination:

$$\begin{aligned} d(x,x) &= 0 \text{ and} \\ d(x,y) &= 0 \rightarrow x = y. \end{aligned}$$

c) Symmetry: For all $x, y \in X$, $d(x, y) = d(y, x)$.
d) Triangle Inequality: For all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

Definition 13 *The pair (X, d), where d is a metric on X, is called a **metric space**.*

Example 14 *The natural metric on $\Re$ is*

$$d(x,y) = |x - y|.$$

*There are several natural metrics on $\Re^M$. The **Euclidean metric** is defined by*

$$d(x,y) = \sqrt{\sum_{i=1}^{M} (x_i - y_i)^2}.$$

The l_1 metric (also called the taxi-cab metric for $\Re^2$) is defined by

$$d(x,y) = \sum_{i=1}^{M} |x_i - y_i|.$$

*The **sup metric** or **uniform metric** is defined by*

$$d(x,y) = \max_{i=1,\dots,M} |x_i - y_i| .$$

Example 15 *Let X be any set and define metric d by $d(x,x) = 0$ and $d(x,y) = 1$ if $x \neq y$. Then, d is a metric on X called the **discrete metric**.*

Exercise Set 2

Exercise 4: Demonstrate that the metrics in the previous two examples satisfy all of the four properties of a metric.

3 Open and closed sets

Definition 16 *An (open) ε-**ball** about $x \in E$, where E is a set in a metric space (X,d), is defined as*

$$B_\varepsilon(x) = \{y \in E : d(y,x) < \varepsilon\}.$$

Definition 17 *A set E in a metric space (X,d) is **open** iff for each $x \in E$ there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq E$. A set is **closed** if its complement is open.*

Definition 18 *A point $x \in X$ is called a **point of closure** of set E in a metric space (X,d) if $\forall \delta > 0\ \exists y \in E$ such that $d(x,y) < \delta$. The set of points of closure of E is denoted by $\overline{E}$ (or $cl(E)$).*

The closure of a set E consists of all points which are intuitively “close to E”.

Theorem 19 *A set E is closed if and only if $E = \overline{E}$.*

Frequently, condition $E = \overline{E}$ is used as a definition of a closed set.

The definition of a point of closure is closely related to the definition of a **limit point:**

Definition 20 *A point $x \in X$ is called a* ***limit point*** *of set E in a metric space (X, d) if $\forall \delta > 0$ $\exists y \in E$ such that $y \neq x$ and $d(x, y) < \delta$.*

The difference between the two definitions is subtle but important — namely, in the definition of the limit point, every open ball about point x must contain a point of the set other than x itself.

Theorem 21 *Every limit point is a point of closure, but not every point of closure is a limit point.*

Definition 22 *A point of closure which is not a limit point is called an* ***isolated point****.*

Theorem 23 *The closure of a set E is equal to the union of E and the set of limit points of E.*

Example 24 *In the natural metric $d(x, y) = |x - y|$ on $\Re$, the ε-ball about $x \in X$ is just the interval $(x - \varepsilon, x + \varepsilon)$. This makes it clear that each interval of the form (a, b) is an open set. There are obviously open sets of forms other than (a, b) (Which are these forms?).*

Theorem 25 *The open sets in a metric space (X, d) have the following properties:*
a) Any union of open sets is open;
b) Any finite intersection of open sets is open;
c) $\varnothing$ and X are open.

Given the definition of a closed set, the corresponding theorem for the closed sets is:

Theorem 26 *The closed sets in a metric space* (X, d) *have the following properties:*
a) Any intersection of closed sets is closed;
b) Any finite union of closed sets is closed;
c) $\varnothing$ *and* X *are closed.*

Definition 27 *The* ***interior*** $Int(E)$ *of set* E *in a metric space* (X, d) *is the open set*

$$Int(E) = \{x \in E : \exists \varepsilon > 0 \text{ such that } d(y, x) < \varepsilon \text{ and } y \in X \text{ imply } y \in E\}.$$

Definition 28 *The* ***boundary*** $Bdry(E)$ *(also denoted* ∂E*) of set* E *in a metric space* (X, d) *is the closed set*

$$Bdry(E) = \overline{E} \setminus Int(E).$$

Every point of a set is either an interior point or a boundary point.

Definition 29 *A set* E *in a metric space* (X, d) *is* ***bounded*** *iff* $E \subseteq B_M(x)$ *for some* $x \in E$ *and* $0 < M < +\infty$.

Definition 30 *A set* E *in the Euclidean space* $\Re^M$ *is* ***compact*** *iff it is closed and bounded.*

Exercise Set 3

Exercise 5: Prove Theorem 19.

Exercise 6: Prove Theorem 21.

Exercise 7: Prove Theorem 23.

Exercise 8: Prove Theorem 25.

Exercise 9: Prove that the interior of any set is open.

Exercise 10: Prove that the boundary of any set is closed.

Exercise 11: Prove that $Int(E) = \left(\overline{E^c}\right)^c$.

4 Infimum and Supremum

Definition 31 *The **greatest lower bound** (or **infimum,** or **inf**) of a subset S of a partially ordered set $(P, \leq)$, denoted as* $\inf(S)$*, is an element l of P such that*
1. $l \leq x$ for all $x \in S$, and
2. For any $p \in P$ such that $p \leq x$ for all $x \in S$ it holds that $p \leq l$.

Definition 32 *The **least upper bound** (or **supremum,** or **sup**) of a subset S of a partially ordered set $(P, \leq)$, denoted as* $\sup(S)$*, is an element u of P such that*
1. $u \geq x$ for all $x \in S$, and
2. For any $p \in P$ such that $p \geq x$ for all $x \in S$ it holds that $p \geq u$.

Definition 33 *(real numbers $\Re$) The **infimum** of a set $S \subseteq \Re$ is $l \in \Re$ such that*
1. $l \leq x$ for all $x \in S$, and
2. $\forall \varepsilon > 0 \; \exists x \in S$ such that $l + \varepsilon > x$.

Definition 34 *(real numbers $\Re$) The **supremum** of a set $S \subseteq \Re$ is $u \in \Re$ such that*
1. $u \geq x$ for all $x \in S$, and
2. $\forall \varepsilon > 0 \; \exists x \in S$ such that $u - \varepsilon < x$.

5 Sequences

Definition 35 *By a **sequence** $\{x_n\}_{n=1}^{\infty}$ from a nonempty set X , we mean an ordered set of the form $(x_1, x_2, ...)$ where $x_i \in X$ for all $i \in \{1, 2, ..., \infty\}$. More formally, a sequence $(x_n)_{n=1}^{\infty}$ is a function from the set of natural numbers N into X. That is, sequence $\{x_n\}_{n=1}^{\infty}$ is the function $f : N \to X$ such that $f(n) = x_n$ for all $n \in N$.*

Definition 36 *Consider a sequence $\{x_n\}_{n=1}^{\infty}$ and a strictly increasing function $\sigma : N \to N$ (that is, $\sigma(k) < \sigma(l)$for any $k, l \in N$ with $k < l$). The sequence $\left\{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, ...\right\} = \left\{x_{\sigma(k)}\right\}_{k=1}^{\infty}$ is called a **subsequence** of sequence $\{x_n\}_{n=1}^{\infty}$.*

Example 37 *Consider sequence* $(0, 1, 0, 1, 0, 1, ...)$. *The subsequence formed by taking the odd-numbered elements of the sequence is given by* $(0, 0, 0, ...)$. *The subsequence formed by taking the even-numbered elements of the sequence is given by* $(1, 1, 1, ...)$.

Definition 38 *Let* (X, d) *be a metric space. A sequence* $\{x_n\}_{n=1}^{\infty}$ ***converges*** *to* $x \in X$, *written* $\lim_{n\to\infty} x_n = x$, *iff* $d(x_n, x)$ *converges to* 0 *as a sequence of real numbers, i.e.* $\lim_{n\to\infty} d(x_n, x) = 0$. *Equivalently,* $\lim_{n\to\infty} x_n = x$ *iff*

$$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ d(x_n, x) < \varepsilon.$$

Definition 39 *(Real numbers R) A sequence of real numbers* $\{x_n\}_{n=1}^{\infty}$ *converges to* $x \in R$, *written* $\lim_{n\to\infty} x_n = x$, *iff* $|x_n - x|$ *converges to* 0 *as a sequence of real numbers, i.e.* $\lim_{n\to\infty} |x_n - x| = 0$. *Equivalently,* $\lim_{n\to\infty} x_n = x$ *iff*

$$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ \ |x_n - x| < \varepsilon.$$

Definition 40 *(*$\Re^M$ *with the Euclidean metric) A sequence* $\{\mathbf{x}_n\}_{n=1}^{\infty}$ *from* $\Re^M$ *(where* $\mathbf{x}_n = (x_n^1, x_n^2, ..., x_n^M)$*) converges to* $\mathbf{x} = (x^1, x^2, ..., x^M) \in \Re^M$, *written* $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$, *iff* $\sqrt{\sum_{i=1}^{M} (x_n^i - x^i)^2}$ *converges to* 0 *as a sequence of real numbers, i.e.* $\lim_{n\to\infty} \sqrt{\sum_{i=1}^{M} (x_n^i - x^i)^2} = 0$. *Equivalently,* $\lim_{n\to\infty} \mathbf{x}_n = \mathbf{x}$ *iff*

$$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ \sqrt{\sum_{i=1}^{M} (x_n^i - x^i)^2} < \varepsilon.$$

6 Continuous Functions

Definition 41 *If* (X, ρ) *and* (Y, σ) *are metric spaces, a function* $f : X \to Y$ *is* ***continuous*** *at* $x_0 \in X$ *iff for each sequence* $\{x_n\}_{n=1}^{\infty}$ *that converges to* x_0, $\lim_{n\to\infty} f(x_n) = f(x_0)$. *Equivalently,* $f : X \to Y$ *is continuous at* $x_0 \in X$ *iff*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ \ [\rho(x, x_0) < \delta \to \sigma(f(x), f(x_0)) < \varepsilon].$$

Definition 42 *(Continuity for a real-valued function on $\Re$ with the natural metric on $\Re$) Let $f : X \to Y$, where $X, Y \subseteq \Re$. $f : X \to Y$ is **continuous** at $x_0 \in X$ iff for each sequence $\{x_n\}_{n=1}^{\infty}$ that converges to x_0, $\lim_{n\to\infty} f(x_n) = f(x_0)$. Equivalently, $f : X \to Y$ is continuous at $x_0 \in X$ iff*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ \left[|x - x_0| < \delta \to |f(x) - f(x_0)| < \varepsilon \right].$$

We can rephrase the definition of a continuous function between metric spaces in terms of open sets in these spaces, thus avoiding explicit mention of the metrics involved:

Theorem 43 *If (X, ρ) and (Y, σ) are metric spaces, a function $f : X \to Y$ is **continuous** at $x_0 \in X$ iff for each open set set V in Y containing $f(x_0)$ there is an open set U in X containing x_0 such that $f(U) \subseteq V$.*

Thus, we can carry the definition of a continuous function to any setting where we can carry a “reasonable” notion of an open set. A “reasonable” definition of an open set is introduced in the following section, where we introduce a mathematical construct that includes metric spaces as a special case.

7 Topological Spaces

Definition 44 *A **topology** on a set X is a collection τ of subsets of X, called the **open sets**, satisfying:*
a) Any union of elements of τ belongs to τ;
b) Any finite intersection of elements of τ belongs to τ;
c) $\varnothing$ and X belong to τ.

Definition 45 *The pair (X, τ), where τ is a topology on X, is called a **topological space.***

Definition 46 *Complements of open sets are called **closed.***

Definition 47 *Let (X, d) be a metric space. Then, by Theorem 25, the open sets defined by Definition 17 form a topology called the* ***metric topology*** *τ_d. Whenever (X, τ) is a topological space whose topology τ is the metric topology τ_d for some metric d on X, we call (X, d) a* ***metrizable topological space****. Every metric space (X, d) defines a metrizable topological space (X, τ_d) and given a metrizable topological space (X, τ), one can always find many metrics d on X such that $\tau_d = \tau$. Two metrics generating the same topology are* ***equivalent****.*

The Euclidean, l_1, and sup metrics on $\Re^n$ are equivalent.

Definition 48 *A property of a metric space that can be expressed in terms of open sets without mentioning a specific metric is called a* ***topological property****.*

There are many topological spaces that are not metrizable.

One can easily introduce a definition of convergence of a sequence in a topological space as well as other notions that you are familiar with for metric spaces. We only provide a definition of continuity of a function:

Definition 49 *If (X, τ) and (Y, υ) are topological spaces, a function $f : X \to Y$ is* ***continuous*** *at $x_0 \in X$ iff for each open set set V in Y containing $f(x_0)$ there is an open set U in X containing x_0 such that $f(U) \subseteq V$.*

Note that this definition is similar to the one given for metric spaces.

8 Lower and Upper Semicontinuity, and Continuity

Definition 50 *A real-valued function $f : X \to \Re$ is* ***upper semicontinuous*** *if for each $a \in \Re$, the upper contour set $\{x : f(x) \geq a\}$ is closed (or equivalently $\{x : f(x) > a\}$ is open). It is* ***lower semicontinuous*** *if every lower contour set $\{x : f(x) \leq a\}$ is closed (or equivalently $\{x : f(x) < a\}$ is open).*

The above definition is valid when X is a general (not necessarily metrizable) topological space.

Theorem 51 *A real-valued function is **continuous** if and only if it is both upper and lower semicontinuous.*

Note that f is upper semicontinuous if and only if $-f$ is lower semicontinuous. We can even talk about semicontinuity at a point. We define it for the case when X is a metric space (One can easily introduce a similar definition for a general topological space.)

Definition 52 *Let (X, d) be a metric space. The real-valued function $f : X \to \Re$ is **upper semicontinuous** at $x \in X$ iff*

$$f(x) \geq \limsup_{y \to x} f(y) = \inf_{\varepsilon > 0} \sup_{0 < d(x,y) < \varepsilon} f(y).$$

*Similarly, f is **lower semicontinuous** at $x \in X$ iff*

$$f(x) \leq \liminf_{y \to x} f(y) = \sup_{\varepsilon > 0} \inf_{0 < d(x,y) < \varepsilon} f(y).$$

*Equivalently, f is **upper semicontinuous** at $x \in X$ iff*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ \left[d(x, y) < \delta \to f(y) < f(x) + \varepsilon \right].$$

*Similarly, f is **lower semicontinuous** at $x \in X$ iff*

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in X \ \left[d(x, y) < \delta \to f(y) > f(x) - \varepsilon \right].$$

9 Correspondences, Lower and Upper Hemicontinuity

Let X and Y be topological spaces. If it is hard for you to think of X and Y as topological spaces, think of them as metric spaces or a specific metric space (for example, the Euclidean space).

Definition 53 *A correspondence* $\varphi : X \rightrightarrows Y$ *associates to each point in* X *a subset of* Y. *For a correspondence* φ, *let*

$$Gr(\varphi) = \{(x, y) \in X \times Y : x \in X, y = \varphi(x)\}$$

denote the ***graph*** *of* φ.

Definition 54 *The* ***upper (or strong) inverse*** *of* E *under* φ *is defined by*

$$\varphi^U(E) = \{x \in X : \varphi(x) \subseteq E\}.$$

The ***lower (or weak) inverse*** *of* E *under* φ *is defined by*

$$\varphi^L(E) = \{x \in X : \varphi(x) \cap E \neq \varnothing\}.$$

Definition 55 *A correspondence* φ *is* ***upper hemicontinuous*** *(uhc) at* x *if whenever* x *is in the upper inverse of an open set, so is a neighborhood of* x. *That is,* φ *is uhc at* x *if* $\forall$*open set* E *with* $x \in \varphi^U(E)$ $\exists$ *open set* A *such that* $x \in A$ *and* $A \subseteq \varphi^U(E)$.

φ *is* ***lower hemicontinuous*** *(lhc) at* x *if whenever* x *is in the lower inverse of an open set so is a neighborhood of* x. *That is,* φ *is lhc at* x *if* $\forall$*open set* E *with* $x \in \varphi^L(E)$ $\exists$ *open set* A *such that* $x \in A$ *and* $A \subseteq \varphi^L(E)$.
The correspondence φ *is upper hemicontinuous (respectively, lower hemicontinuous) if it is upper hemicontinuous (respectively, lower hemicontinuous) at every* $x \in X$. *Thus,* φ *is upper hemicontinuous (respectively, lower hemicontinuous) if the upper (respectively, lower hemicontinuous) inverses of open sets are open.*

A correspondence is ***continuous*** *if it is both upper and lower hemicontinuous.'*

When φ is compact-valued the above definitions of hemicontinuity are equivalent to the following:

Definition 56 *A compact-valued correspondence* φ *is uhc at* x *if* $\forall \{x_n\}_{n=1}^{\infty}$ *with* $\lim_{n\to\infty} x_n = x$ *and* $\forall \{y_n\}_{n=1}^{\infty}$ *with* $y_n \in \varphi(x_n)$ *for all* n $\exists$ *a convergent subsequence* $\{y_{k_n}\}_{n=1}^{\infty}$ *of* $\{y_n\}_{n=1}^{\infty}$ *such that* $\lim_{n\to\infty} y_{k_n} \in \varphi(x)$.

A compact-valued correspondence φ *is lhc at* x *if* $\forall y \in \varphi(x)$ *and* $\forall \{x_n\}_{n=1}^{\infty}$ *with* $\lim\limits_{n\to\infty} x_n = x$ $\exists N \geq 1$ *and a sequence* $\{y_n\}_{n=N}^{\infty}$ *such that* $\lim\limits_{n\to\infty} y_n = y$ *and* $y_n \in \varphi(x_n)$ *for all* $n \geq N$.

If correspondence φ is singleton-valued, the upper and the lower inverses of a set coincide and agree with the inverse regarded as a function. Either form of hemicontinuity is equivalent to the continuity of a function. The term "semicontinuity" has been used to mean hemicontinuity, but this usage can lead to confusion when discussing real-valued singleton correspondences. A semicontinuous real-valued function is not a hemicontinuous correspondence unless it is also continuous.

Definition 57 *A correspondence* φ *is* ***closed*** *at* x *if* $\forall \{x_n\}_{n=1}^{\infty}$ *with* $\lim\limits_{n\to\infty} x_n = x$ *and* $\forall \{y_n\}_{n=1}^{\infty}$ *with* $y_n \in \varphi(x_n)$ *for all* n*, then* $y \in \varphi(x)$*. A correspondence is closed if it is closed at every point of its domain, that is, if its graph is closed.*

In general, a correspondence may be closed without being upper hemicontinuous, and vice versa. On the other hand, we have:

Theorem 58 *Let* $X \subseteq \Re^m$ *and* $Y \subseteq \Re^k$*. Then*
1. If φ *is upper hemicontinuous and closed-valued, then* φ *is closed.*
2. If Y *is compact and* φ *is closed, then* φ *is upper hemicontinuous.*
3. If φ *is singleton-valued at* x *and upper hemicontinuous at* x*, then* φ *is continuous at* x*.*

10 References:

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