Techniques of Problem Solving

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CHAPTER 1

Strategy-tactics

1. Pursue parity

Problem 1.1. Let \( n \) be a an odd integer greater than 1 and let \( A \) be a \( n \times n \) symmetric matrix such that each row and each column consists of some permutation of the integers \( 1, 2, 3, \ldots, n \). Show that each one of these integers must appear in the main diagonal of \( A \).

Problem 1.2. Place a knight on each square of a \( 7 \times 7 \) chessboard. Is it possible for each knight to simultaneously make a legal move?

Solution : Initially 25 knight are on white squares while 24 knights are on black squares. If we assume it is possible, then all of them will change color and we will end up with 25 knight on black squares. Contradiction.

Problem 1.3. A \( 8' \times 8' \) bathroom is to have its floor tiled. Each tile is \( 2' \times 1' \). In one corner of the bathroom is a sink, and its plumbing occupies a \( 1' \times 1' \) square in the floor. In the opposite corner is a toilet, and its plumbing occupies a \( 1' \times 1' \) square in the floor. How is it possible to achieve the required tiling of the floor?

Solution : Look at the bathroom floor like a chess table. Then 32 squares are black and 30 squares are white. But each \( 2 \times 1 \) tile has the same number of white and black squares.

Problem 1.4. Imagine a polyhedron with 13 vertices, and imagine that each edge is assigned an electrical charge of +1 or -1. Prove there must be a vertex such that the product of the charges of all the edges that meet at that vertex is +1.

Problem 1.5. Consider a society of \( n \) people.

a) Prove that the number of people with an odd number of friends is even.

b) If everyone has exactly 3 friends, prove that \( n \) is even.

Solution : a) Consider a graph with the set of vertices consisting of the people, and if \( A \) is a friend of \( B \) then we have the edge \((A,B)\). Of course, we have also the edge \((B,A)\), so we have an even number of edges. But the total number of edges is the sum of the number of friends of each person, which ends the proof.

b) Application of a), since we have \( n \) people with an odd number of friends.

Problem 1.6. A rook stands on the lower left square of a chessboard. Is there a path that takes the rook through every square of the chessboard once and only once, and that ends at the upper right square?

Solution : No such path exists. If the squares are alternately black and white, the beginning and ending squares are the same color. Any path can be broken up into
short paths that advance only one square, which changes color each time. Since 63 moves are required, the change of color occurs an odd number of times, leaving the rook on the opposite color from which it started.

**Problem 1.7.** Let $S$ be a set of 1003 integers such that the sum of any two of them is divisible by 1003. Prove that all the numbers in $S$ are divisible by 1003.

**Solution:** Let $S$ be a set of 1003 integers such that the sum of any two of them is divisible by 1003. Prove that all the numbers in $S$ are divisible by 1003.

**Problem 1.8.** Put a knight on a $7 \times 7$ chessboard. Is it possible, in 49 consecutive knight moves, to visit each square exactly once and return to the original square?

**Solution:** At each move the knight changes the color of the square. After 49 moves the knight will be on a square with a different color than the original one, so the answer is no.

**Problem 1.9.** A knight stands on the lower left square of a chessboard. Is there a path that takes the knight through every square of the chessboard once and only once, and that ends at the upper right square?

**Solution:** No such path exists. If the squares are alternately black and white, the beginning and ending squares are the same color. At each move the square changes the color. Since 63 moves are required, the change of color occurs an odd number of times, leaving the knight on the opposite color from which it started.

**Problem 1.10.** Let $P_1, P_2, \ldots, P_{11}$ be 11 points in the plane, no three collinear, joined by the 13 segments $P_1P_2$, $P_2P_3$, $\ldots$, $P_{10}P_{11}$, $P_{11}P_1$. Can these points be chosen in such a way that there is a line which crosses all 11 segments and is not meeting any of the points $P_i$?

**Solution:** Suppose there is such a line $L$. Since $L$ is crossing $P_iP_{i+1}$, the points $P_i$ and $P_{i+1}$ are on different semi-planes, for any $i = 1, 2, \ldots, 10$. But then the points with the odd indices are in the same semiplane, so $P_{11}P_1$ cannot cross $L$.

**Problem 1.11.** 23 people decide to play football: 11 people on each team plus one referee. To keep things fair, they agree that the total weight of each team must be the same. Everyone weighs an integer, and it turns out that no matter who is chosen to be referee, it is always possible to construct two fair (sums of weighs are equal) teams. Prove that everyone weighs the same.

**Solution:** Let $W = \{w_1, w_2, \ldots, w_{23}\}$ be the weights. This set has the property $P$: the set obtained after we remove any element can be partitioned as reunion of two sets of 11 elements with the same sum. The elements of $W$ have the same parity since $\sum_{k=1}^{23} w_k - w_i$ is even for any $i$. We can suppose without loss of generality that $w_1 \leq w_2 \leq \ldots \leq w_{23}$. Let $x_i = w_i - w_1$. The set $X = \{x_1, x_2, \ldots, x_{23}\}$ has also the property $P$, so all of its elements have the same parity with $x_1 = 0$, hence they are even. Let $y_i = x_i/2$, for any $i$. The set of integers $Y = \{y_1 = 0, y_2, \ldots, y_{23}\}$ has also the property $P$. Using the infinite descent method we obtain $x_1 = x_2 = \ldots = x_{23} = 0$.

**Problem 1.12.** Put a knight on a $4 \times 8$ chessboard. Is it possible, in 32 consecutive moves, to visit each square once and to return to the original square?
2. Argue by contradiction

**Problem 1.13.** Let \( a > 0 \), \( n \) a positive integer, \( x_1, x_2, \ldots, x_n \in [0, a] \), and \( \phi \) a permutation of the set \( \{1, 2, \ldots, n\} \). Prove there is a \( j \in \{1, 2, \ldots, n\} \) such that
\[
x_j(a - x_{\phi(j)}) \leq \frac{a^2}{4}.
\]

**Solution:** If we suppose the contrary, then for any \( i \in \{1, 2, \ldots, n\} \) we have
\[
x_i(a - x_{\phi(i)}) > \frac{a^2}{4}.
\]

Multiplying all these inequalities we obtain
\[
\left(\frac{a^2}{4}\right)^n < \prod_{i=1}^{n} x_i(a - x_{\phi(i)}) = \prod_{i=1}^{n} x_i(a - x_i).
\]

By the inequality of means, for each \( i \in \{1, 2, \ldots, n\} \) we have
\[
x_i(a - x_i) \leq \left(\frac{x_i + (a - x_i)}{2}\right)^2 = \frac{a^2}{4}
\]

We multiply all these inequalities and we obtain
\[
\prod_{i=1}^{n} x_i(a - x_{\phi(i)}) \leq \left(\frac{a^2}{4}\right)^n
\]

Contradiction!

**Problem 1.14.** Prove that the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent.

**Solution:** Assume the series is convergent to \( s \in \mathbb{R} \). We observe that
\[
\frac{1}{2n-1} \geq \frac{1}{2n}
\]
for any \( n \geq 2 \). Then we have
\[
s = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \ldots
\]
\[
= 1 + \left(\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \ldots\right) + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{8} + \ldots\right)
\]
\[
\geq 1 + \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots\right) + \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \frac{1}{8} + \ldots\right)
\]
\[
= \frac{1}{2} + 2 \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \ldots\right) = \frac{1}{2} + s
\]

Contradiction!

**Problem 1.15.** Prove there are infinitely many prime integers.

**Solution:** Assume there exist only finitely many prime numbers \( p_1, p_2, p_3, \ldots, p_n \). Consider the number \( N = p_1p_2p_3\ldots p_n + 1 \) which is relatively prime with each of the numbers \( p_1, p_2, \ldots, p_n \). Then \( N \) is either prime or is the product of other primes not in the above list. Contradiction!

**Problem 1.16.** Let \( n \) be a positive integer. Prove that one of the integer numbers \( n, n+1, n+2, \ldots, 2n-1, 2n \) is a perfect square.
Solution: Assume there is a positive integer $n$ such that none of the numbers
$n, n+1, n+2, \ldots, 2n$ is a perfect square. Consider $k^2$ the greatest perfect square
smaller than $n$. Then $k^2 \leq n - 1$ and $2n + 1 \leq (k+1)^2$ and from here

$$k^2 + 1 \leq n \leq \frac{k^2 + 2k}{2}$$

But the inequality $k^2 + 1 \leq \frac{k^2 + 2k}{2}$ can be written $(k-1)^2 \leq -1$. Contradiction!

Problem 1.17. Let $a, b, c$ be odd integers. Prove that the equation $ax^2 + bx + c = 0$
cannot have rational solutions.

Solution: Assume the equation has a rational solution $\frac{p}{q}$ which leads to the
relation

$$ap^2 + bpq + cq^2 = 0$$

Simplifying if needed, we can assume that $p$ and $q$ are relatively prime numbers.
We distinguish the following cases:

Case $p$ even, $q$ odd. Then $ap^2 + bpq + cq^2$ is odd, hence non-zero.
Case $p$ odd, $q$ even. Then $ap^2 + bpq + cq^2$ is odd, hence non-zero.
Case $p$ odd, $q$ odd. Then $ap^2 + bpq + cq^2$ is odd, hence non-zero.

Each of the possible cases leads to a contradiction which shows our assumption
is false.

Problem 1.18. Prove that there is no function $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x) + f(1-x) = x,$$
for any real $x$.

Solution: Assume there is function $f$ such that for any $x$ real we have

$$f(x) + f(1-x) = x.$$

In particular, for $x = 0$ we have $f(0) + f(1) = 0$ and for $x = 1$ we have $f(1) + f(0) = 1$. Contradiction!

Problem 1.19. Prove that if $2^n + 1$ is a prime number, then the positive integer
$n$ is a power of 2.

Solution: Assume there is $n$ not a power of 2 for which $2^n + 1$ is a prime number. Then there is $k$ positive integer and $m$ integer such that $n = 2^m(2k + 1)$. In the formula

$$a^{2k+1} + b^{2k+1} = (a+b)(a^{2k} - a^{2k-1}b + a^{2k-2}b^2 - \ldots + a^2b^{2k-2} - ab^{2k-1} + b^{2k})$$
we take $a = 2^m$ and $b = 1$ and we see that $2^n + 1 = (2^m)^{2k+1} + 1$ is divisible by $2^{2m} + 1$. Contradiction!

Problem 1.20. Prove there is no arithmetic progression which has $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$
among its terms.

Solution: Assume there is an arithmetic progression $a_1, a_2, a_3, \ldots$ with ratio $r$
which has the above three terms. There exist then positive integers $m, n, p$ such
that $a_m = \sqrt{2}$, $a_n = \sqrt{3}$ and $a_p = \sqrt{5}$. Then $\sqrt{3} - \sqrt{2} = (m-n)r$ and $\sqrt{5} - \sqrt{3} = \ldots$
(p−n)r, so \( \frac{\sqrt{5} - \sqrt{3}}{\sqrt{3} - \sqrt{2}} = \frac{p-n}{n-m} \) is rational. But \( \frac{\sqrt{5} - \sqrt{3}}{\sqrt{3} - \sqrt{2}} = (\sqrt{5} - \sqrt{3})(\sqrt{3} + \sqrt{2}) = \sqrt{15} - 3 + \sqrt{10} - \sqrt{6} \) is a rational number. From here we get \((a + \sqrt{6})^2 = (\sqrt{15} + \sqrt{10})^2 \) and in a next step \((2a - 10)\sqrt{6} = 19 - a^2 \). We see that \( a = 5 \) does not satisfy the equality, so we divide by \( 2a - 10 \) and have 

\[
\sqrt{6} = \frac{19 - a^2}{2a - 10} \in \mathbb{Q}.
\]

Contradiction!

**Problem 1.21.** Let \( a, b, c \) be rational numbers such that \( a + b\sqrt{2} + c\sqrt{4} = 0 \).
Prove that \( a = b = c = 0 \).

**Problem 1.22.**
(a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?
(b) What if “three” is replaced by “nine”?

**Solution:**
(a) The answer is yes. Let us suppose the contrary. If \( A \) is a fixed point in the plane then all the points situated at the distance \( \sqrt{3} \) from \( A \) have the same color like \( A \). Indeed, if \( ABC \) and \( BCD \) are equilateral triangles with the length of the side equal 1 inch, then \( A \) and \( D \) are of the same color (the three points of such an equilateral triangle must have different colors) and for any point \( D \) situated at the distance \( \sqrt{3} \) from \( A \) there are points \( B \) and \( C \) such that \( ABC \) and \( BCD \) are equilateral triangles with the length of the side equal 1 inch. But on the circle with the center \( A \) and radius \( \sqrt{3} \) one can find two points at the distance 1 inch.

(b) In this case the answer is no. It suffices to make a partition of the plane in squares of side \( \frac{3}{2} \), then partition this squares in 9 little squares of side \( \frac{1}{2} \), all nine of different colors, and with the same arrangement of colors in all the “big” squares.

### 3. Work backward

**Problem 1.23.** There are 10 cities in a state, and some pairs of cities are connected by roads. There are 40 roads altogether. A city is a hub if it is directly connected to every other city. What is the largest possible number of hubs?

**Problem 1.24.** Around a circle, 5 ones and 4 zeros are arranged in any order. Then between any two equal digits, you write 0 and between different digits 1. Finally, the original digits are wiped out. If this process is repeated indefinitely, you can never get 9 zeros. Generalize.

**Problem 1.25.** If \( a, b, c \) are lengths of three segments which can form a triangle, show that the same, is true for \( \frac{1}{a+c}, \frac{1}{a+b}, \frac{1}{b+c} \).

**Problem 1.26.** Two players named A and B are playing the following game: A is choosing a integer number between 1 and 10, B adds to A’s number a number between 1 and 10, then each player at his turn adds to the current number a number between 1 and 10. The winner is the player who says 100. The following sequence is an example of numbers chosen successively by A and B: 3, 8, 15, 23, 30, 37, 43, 51, 59, 60, 69, 79, 82, 87, 93, 100. Show that the first player can always win the game. Generalization.
Solution: If one of the players is choosing 89 then he will win because independently of the choice of the other player he will be able to choose 100. Also the player we is choosing 78 will be able to choose 89, so he wins. Going back in this way we see that the winning numbers are 1, 12, 23, 34, 45, 56, 67, 78, 89, 100. So the first player will win if he starts with 1.

One can generalize with \( n \) instead of 100 and \( m \) instead of the step 10. If \( q \) and \( r \) are the quotient and the remainder of the division of \( n \) by \( m + 1 \) then \( n = (m + 1)q + r \). The first player can win if he starts by \( r \) and after he will choose the winning numbers \( r + (m + 1), r + 2(m + 1), \ldots, r + (q - 1)(m + 1), r + q(m + 1) = n \).

Problem 1.27. (Infinite descent) Solve in integers the equation \( x^2 + y^2 + z^2 = x^2y^2 \).

Problem 1.28. (Infinite descent) Solve in integers the equation \( x^3 + 12y^3 = 10z^3 \).

4. Use effective notation

Some useful notations are \( \sum, \prod \), and \( \overline{ab} = 10a + b \) for \( a,b \) digits with \( a \neq 0 \).

Problem 1.29. Solve the equation \( \left( x^3 - \frac{9}{16}x \right)^3 - \frac{9}{16} \left( x^3 - \frac{9}{16}x \right) = x \).

Solution: Denote \( x^3 - \frac{9}{16}x = y \). We have then the system

\[
\begin{align*}
x^3 - \frac{9}{16}x &= y \\
y^3 - \frac{9}{16}y &= x
\end{align*}
\]

Subtracting the two equations we obtain

\[
(x - y) \left( x^2 + xy + y^2 + \frac{7}{16} \right) = 0
\]

The expression in the second factor of the left hand side is always positive, hence \( x - y = 0 \). We substitute in any of the original equations of the system to obtain \( x^3 - \frac{25}{16}x = 0 \), equation with the solutions \( x_1 = 0 \), \( x_2 = \frac{5}{4} \), and \( x_3 = -\frac{5}{4} \).

Problem 1.30. Determine the consecutive integers with the property that their sum is 2018.

Solution: Denote by \( n \) the number of consecutive integers and by \( x \) the first of them. The numbers are \( x, x + 1, x + 2, \ldots, x + n - 2, x + n - 1 \).

The sum of these numbers is

\[
\sum_{k=0}^{n-1} (x + k) = \sum_{k=0}^{n-1} x + \sum_{k=0}^{n-1} k = nx + \frac{(n - 1)n}{2} = \frac{n(2x + n - 1)}{2}
\]
We have to find now the integers \( n > 0 \) and \( x \) such that
\[
n(2x + n - 1) = 2 \cdot 2018 = 4 \cdot 1009
\]
Let us observe that \((2x + n - 1) - n = 2x - 1\) is odd, hence the numbers \( n \) and \( 2x + n - 1 \) have different parities. Also we check that 1009 is a prime number. We distinguish then the following cases

I. \( n = 1, 2x + n - 1 = 4036 \). Then \( n = 1, x = 2018 \) and the sequence consists of only one number 2018

II. \( n = 4, 2x + n - 1 = 1009 \). Then \( n = 4, x = 503 \) and the sequence consists of 503, 504, 505, 506.

III. \( n = 1009, 2x + n - 1 = 4 \). Then \( n = 1009, x = -502 \) and the sequence consists of
\[-502, -501, -500, \ldots, 505, 506\]

IV. \( n = 4036, 2x + n - 1 = 1 \). Then \( n = 4036, x = -2017 \) and the sequence consists of
\[-2017, -2016, -2015, \ldots, 2017, 2018\]

**Problem 1.31.** a) If \( n \) is a positive integer such that \( 2n + 1 \) is a perfect square, show that \( n + 1 \) is the sum of two successive perfect squares.

b) If \( 3n + 1 \) is a perfect square, show that \( n + 1 \) is the sum of three perfect squares.

**Solution:**

a) Since \( 2n + 1 \) is odd and a perfect square, there is an integer \( k \) such that \( 2n + 1 = (2k + 1)^2 \) which corresponds to \( n = 2k^2 + 2k \). Then
\[
n + 1 = k^2 + (k + 1)^2
\]

b) Since \( 3n + 1 \) is a perfect square there is an integer \( k \) such that \( 3n + 1 = (3k \pm 1)^2 \) which corresponds to \( n = 3k^2 \pm 2k \). Then
\[
n + 1 = k^2 + k^2 + (k \pm 1)^2
\]

**Problem 1.32.** Let \(-1 < a_0 < 1\) and define recursively \( a_n = \sqrt{1 + a_{n-1}} \), \( n > 0 \). Let \( A_n = 4^n(1 - a_n) \). What happens to \( A_n \) as \( n \) tends to infinity?

**Solution:** Denote \( \alpha = \arccos a_0 \in (0, \pi) \). We prove by induction that \( a_n = \cos \frac{\alpha}{2^n} \) and then
\[
A_n = 2^n \left( 1 - \cos \frac{\alpha}{2^n} \right) = 2^{n+1} \sin \frac{\alpha}{2^{n+1}} \xrightarrow{n\to\infty} \alpha
\]

**Problem 1.33.** A car travels from A to B at the rate of 40 miles per hour and then from B to A at the rate of 60 miles per hour. Is the average rate for the trip more or less than 50 miles per hour?
Solution: Denote by $d$ the distance between A and B, by $t_1 = \frac{d}{40}$ the time in which the car travelled from A to B, and by $t_2 = \frac{d}{60}$ the time in which the car travelled from B to A. Then the average speed for the whole travel is

$$v = \frac{d + d}{t_1 + t_2} = \frac{2d}{\frac{d}{40} + \frac{d}{60}} = 48 \text{ miles per hour}$$

Problem 1.34. You are given a cup of coffee and a cup of cream, each containing the same amount of liquid. A spoonful of cream is taken from the cup and put into the coffee cup, then a spoonful of the mixture is put back into the cream cup. Is there now more or less cream in the coffee cup than coffee in the cream cup?

Problem 1.35. Find the sum of the reciprocals of all the positive divisors of 144.

Solution: We observe that $144 = 2^4 \cdot 3^2$ and that any divisor of 144 is of the form $2^i \cdot 3^j$ where $0 \leq i \leq 4$ and $0 \leq j \leq 2$. Thus the sum we look for is

$$\sum_{i=0}^{4} \sum_{j=0}^{2} \frac{1}{2^i} \cdot \frac{1}{3^j} = \left( \sum_{i=0}^{4} \frac{1}{2^i} \right) \left( \sum_{j=0}^{2} \frac{1}{3^j} \right) = \frac{1 - \frac{1}{2^5}}{1 - \frac{1}{2}} \cdot \frac{1 - \frac{1}{3^3}}{1 - \frac{1}{3}} = \frac{409}{144}$$

Problem 1.36. In a village, $2/3$ of the men are married to $3/4$ of the women. What fraction of the adult population is married? We assume that in this village a marriage consists of an union between one man and one woman.

Solution: Denote by $M$ the number of men and by $W$ the number of women in the village. We know that $\frac{2M}{3} = \frac{3W}{4}$, so $M = \frac{9W}{8}$. Then the proportion of married people in the adult population is

$$\frac{2M + 3W}{M + W} = \frac{3W + 3W}{9W + 8W} = \frac{12}{17}$$

Problem 1.37. Solve the equation $\sqrt{x} + \sqrt{y - 1} + \sqrt{z - 2} = \frac{x + y + z}{2}$.

Solution: We will introduce new variables to get an equation without radicals. So we denote $\sqrt{x} = u, \sqrt{y - 1} = v, \sqrt{z - 2} = w$ and we will substitute $x = u^2, y = v^2 + 1, z = w^2 + 2$ in the equation which becomes

$$(u - 1)^2 + (v - 1)^2 + (w - 1)^2 = 0$$

A sum of squares is zero only when each of the terms is zero, so $u = v = w = 1$ and from here $x = 1, y = 2$, and $z = 3$. 
Problem 1.38. Evaluate the following
\[(2 \cdot 5 + 2)(4 \cdot 7 + 2)(6 \cdot 9 + 2)\ldots(1998 \cdot 2001 + 2)\]
\[(1 \cdot 4 + 2)(3 \cdot 6 + 2)(5 \cdot 8 + 2)\ldots(1997 \cdot 2000 + 2)\]

Solution: The expression is
\[\prod_{n=1}^{999} \frac{2n(2n + 3) + 2}{(2n - 1)(2n + 2) + 2} = \prod_{n=1}^{999} \frac{(2n + 1)(n + 1)}{n(2n + 1)} = 1000\]

5. Search for a pattern

Problem 1.39. You are given 2 marbles and the task of determining the highest story of a 100 story building from which you can drop a marble without breaking it. Find a scheme that is guaranteed to give the answer in the fewest number of drops.

Problem 1.40. The cells in a jail are numbered from 1 to 100 and their doors are activated from a central button. The activation opens a closed door and closes an open door. Starting with all the doors closed the button is pressed 100 times. When it is pressed \(k\)-th time the doors multiple of \(k\) are activated. Which doors will be open at the end.

Solution: Denote by \(\langle k \rangle\) the number of divisors of \(k\). Then the door \(k\) will be activated \(\langle k \rangle\) times. At the end will be open the doors having an odd number of divisors. Let us notice that \(\langle k \rangle\) is odd if and only if \(k\) is a perfect square. Indeed the divisors of \(k\) can be arranged in pairs \(\left(\frac{k}{n}, n\right)\), excepting the case when \(n = \frac{k}{n}\), or \(k = n^2\). So at the end the doors 1, 4, 9, \ldots, 100 will be open.

Problem 1.41. Show that for any integer \(n > 1\), the number \(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\) is not an integer.

Problem 1.42. Prove that the sum of 5 consecutive perfect squares cannot be a perfect square.

Solution: \((n - 2)^2 + (n - 1)^2 + n^2 + (n + 1)^2 + (n + 2)^2 = 5(n^2 + 2)\) is divisible by 5 but not divisible by 25, so it cannot be a perfect square.

Problem 1.43. Prove that for any positive integer \(n\), the number \(2 \cdot 3^{4n-1} + 1\) is not a prime.

Problem 1.44. Let \((x_n)_n\) be a sequence defined by \(x_0 = a\) and \(x_{n+1} = \frac{x_n - 1}{x_n + 1}\) for any integer \(n \geq 0\). If \(x_{2006} = 3\), find \(a\).

Solution: Let \(f(x) = \frac{x - 1}{x + 1}\). Then \(x_{n+1} = f(x_n)\). Since \(f(f(x)) = -\frac{1}{x}\) and \((f \circ f \circ f \circ f)(x) = x\), the sequence \((x_n)_n\) is periodic of period 4. Then \(x_{2006} = x_2 = -\frac{1}{a}\), so \(a = -\frac{1}{2006}\).
Problem 1.45. Given a positive integer \( n \) determine the minimum number \( k \) of weights needed to weight out any whole number of pounds from 1 to \( n \). The weight can be used by putting one or more of these weights on one pan of a scale.

Problem 1.46. Determine the number of matches at a tennis tournament of \( n \) players.

Problem 1.47. How many diagonals does a polygon of \( n \) sides have?

Problem 1.48. Determine the number of subsets of a set with \( n \) elements.

Problem 1.49. In how many ways can the positive integer \( n \) be written a as a sum of two nonzero positive integers?

6. Modify the problem. Formulate an equivalent problem

Problem 1.50. Solve the system
\[
\begin{align*}
2(x + y) + xy &= 5 \\
2(y + z) + yz &= 21 \\
2(z + x) + zx &= 12
\end{align*}
\]

Solution: The system can be written
\[
\begin{align*}
(x + 2)(y + 2) &= 9 \\
(y + 2)(z + 2) &= 25 \\
(z + 2)(x + 2) &= 16
\end{align*}
\]

Multiplying the equations we have \([(x + 2)(y + 2)(z + 2)]^2 = 3^2 \cdot 5^2 \cdot 4^2\), etc.

Problem 1.51. Let \( P \) be a polynomial and \( k \) an odd positive integer. Prove that \( P \) is reciprocal if and only if \( P^k \) is reciprocal.

Solution: We need to use the following characterization of reciprocal polynomials:

Let \( P \) be a polynomial of degree \( n \). Then \( P \) is reciprocal if and only if
\[
x^n P \left( \frac{1}{x} \right) = P(x), \text{ for any } x \neq 0
\]

Problem 1.52. Let \( (x_n)_n \) be a sequence of real numbers. Prove that \( \lim_{n \to \infty} \left( 1 + \frac{1}{x_n} \right)^{x_n} = e \) if and only if \( \lim_{n \to \infty} x_n (\ln(1 + x_n) - \ln x_n) = 1 \).

Problem 1.53. Find the minimum value of the set
\[
A = \left\{ a \in \mathbb{R} \mid \cos x > \frac{1}{x + a}, \forall x \in (0, 1) \right\}.
\]

Problem 1.54. Prove that for any positive integer \( n \) we have the inequality
\[
\sqrt{1 + \frac{\sqrt{n}}{n}} + \sqrt{1 - \frac{\sqrt{n}}{n}} < 2.
\]

Solution: Consider the function \( f(x) = \sqrt{1 + x} \). It suffices to prove that \( f \) is a concave function.

Problem 1.55. Evaluate the sum \( 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + ... + n \cdot 2^n \).
Solution: Determine a closed form for the function
\[ f(x) = x + 2x^2 + 3x^3 + \ldots + nx^n. \]

Problem 1.56. For \( a > 1 \) let \( M_a = \{ x > 0 \mid x^a = a^x \} \). Determine \( a \) such that \( M_a \) has exactly one element.

Problem 1.57. If \( a \) and \( b \) are integers such that \( a + \sqrt{b} = \sqrt{15 + \sqrt{216}} \) find \( a/b \).

Solution: Using the nested radicals formula one can determine the values of \( a \) and \( b \).

Problem 1.58. A teacher must divide 221 apples evenly among 403 students. What is the minimal number of pieces into which she must cut the apples?

Problem 1.59. Prove that \( \sqrt{20 + 14\sqrt{2}} + \sqrt{20 - 14\sqrt{2}} = 4 \).

Problem 1.60. Let \( A \) be a set with \( n \) elements. How many solutions has the equation \( A = X \cup Y \)?

Problem 1.61. Ask a friend to pick a number from 1 to 1000. After asking him 10 questions that can be answered yes or no, you tell him the number. What kind of questions?

7. Divide and conquer

Problem 1.62. Let \( 0 < x_1 < x_2 < \ldots x_{10} \) be integers such that \( x_1 + x_2 + \ldots + x_{10} = 60 \).

Prove the following
i) There are either 4 or 6 even numbers among these 10 numbers.
ii) \( x_{10} \leq 15 \)
iii) \( x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5 \)
iv) The product \( x_1x_2\ldots x_{10} \) is divisible by 1440.
v) How many such groups of 10 numbers are there?

Solution: i) Since the sum is even, the number of odd terms in the sum is even. Then we also have an even number of even terms. We’ll prove now that we have at least 4 odd terms. Otherwise, if we have at most 2 odd terms the smallest sum we can make is
\[ x_1 + \ldots + x_{10} \geq 1 + 2 + 3 + 4 + 6 + 8 + 10 + 12 + 14 + 16 = 76 > 60 \]
It suffices now to show that we have at least 4 even terms in the sum. Indeed, if there are at most 2 even terms the smallest sum we can make is
\[ x_1 + \ldots + x_{10} \geq 1 + 2 + 3 + 4 + 5 + 7 + 9 + 11 + 13 + 15 = 70 > 60 \]
i) If \( x_{10} > 16 \), then
\[ x_1 + x_2 + \ldots + x_9 + x_{10} \geq 1 + 2 + \ldots + 9 + 16 = 61 \]
ii) If \( x_5 \geq 5 \), then the sum is at least
\[ x_1 + x_2 + \ldots + x_9 + x_{10} \geq 1 + 2 + 3 + 4 + 6 + 7 + 8 + 9 + 10 + 11 = 61 \]
From $x_5 = 5$, the other equalities follow immediately.

iv) The first 5 numbers have a product of 120. We need to prove that the product of the other 5 numbers is divisible by 12 = $2^2 \cdot 3$. Since between the first 5 numbers we have two even, there are at least 2 more even numbers between the remaining 5, so their product is divisible by $2^2$. If we assume none of these 5 numbers is divisible by 3, then the sum of all 10 is at least

$$x_1 + \ldots + x_{10} \geq 1 + \ldots + 5 + 7 + 8 + 10 + 11 + 13 = 64 > 60$$

v) An analysis of the cases shows there are 7 such groups of 10 positive integers

1. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 15 = 60$
2. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 10 + 14 = 60$
3. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 11 + 13 = 60$
4. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 9 + 10 + 13 = 60$
5. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 9 + 11 + 12 = 60$
6. $1 + 2 + 3 + 4 + 5 + 7 + 8 + 9 + 10 + 11 = 60$

Problem 1.63. Determine the non-negative integers $n, a, b, c$ such that

$$2^n = a! + b! + c!$$

Problem 1.64. For which integer values of $n$ there exist integers $x, y$ such that

$$x^2 - y^2 = n.$$ 

Problem 1.65. Determine the positive integer $n$ and the digits $a_1, a_2, \ldots, a_n$ such that

$$\sqrt{a_1a_2\ldots a_{n-1}a_n} - \sqrt{a_1a_2\ldots a_{n-2}} = a_n$$

Problem 1.66. How many 5-digit integer numbers have the product of the digits equal to 180?

Problem 1.67. Find the minimum possible value of the largest of $xy, 1 - x - y + xy,$ and $x + y - 2xy$ if $0 \leq x \leq y \leq 1$.

Problem 1.68. One of the receipts for a Math tournament showed that 72 identical trophies were purchased for 99.9, where the first and the last digits were illegible. How much did each trophy cost?

Problem 1.69. Knowing that $m, n$ are integers such that $3m \geq 2n - 3$ and $4n \geq m + 12$ determine the smallest possible value of $\frac{m}{n}$.
CHAPTER 2

Important principles

1. Invariance principle

Problem 2.1. Let \( P(x) = x^3 - x^2 + 1 \) and consider the transformations \( P(x) \mapsto 3P(x)^3 - 2 \) and \( P(x) \mapsto P^2(x) - P(x) + 1 \). Is it possible, applying to \( P \) these transformations, in any order, to get \( Q(x) = x^{24n} + x^{12n} - x^{3n} + n^2 + 3n + 6 \), for some positive integer \( n \)?

Solution: The answer is negative, since \( P(1) = 1 \) and the transformations are preserving this property.

Problem 2.2. Consider 5 points in the plane. What are the possible values of the total number of lines they determine?

Solution: We examine the cases by the maximum number of points out of the 5 which are on the same line. Then the possible values are 1 when all are collinear, 5 when 4 are collinear, 6 when 3 are collinear and other 3 are collinear with one point common to these two groups of three points, 8 when 3 are collinear and the line determined by the other two doesn’t contain any of the initial 3, and finally 10 lines when the points are on a circle for example.

Problem 2.3. Consider 10 distinct points in the plane. Is it possible that these points determine exactly 44 lines in the planes? What about 43, 40, 41, or 42?

Problem 2.4. If \( a_1, a_2, \ldots, a_n \) are integers, prove that \( |a_1 - a_2| + |a_2 - a_3| + \ldots + |a_{n-1} - a_n| + |a_n - a_1| \) and \( |a_1 + a_2| + |a_2 + a_3| + \ldots + |a_{n-1} + a_n| + |a_n + a_1| \) are even numbers.

Solution: The key observation is that the integer \( n \) has the same parity with the integer \( |n| \). Then \( |a_1 - a_2| + |a_2 - a_3| + \ldots + |a_{n-1} - a_n| + |a_n - a_1| \) has the same parity with the number \( a_1 - a_2 + a_2 - a_3 + \ldots + a_{n-1} - a_n + a_n - a_1 = 0 \) and \( |a_1 + a_2| + |a_2 + a_3| + \ldots + |a_{n-1} + a_n| + |a_n + a_1| \) has the same parity with \( a_1 + a_2 + a_2 + a_3 + \ldots + a_{n-1} + a_n + a_n + a_1 = 2(a_1 + a_2 + \ldots + a_n) \).

Problem 2.5. There are 2002 plastic disks on a table and a large supply of extra disks. Some of the disks are colored red, some white and some blue. At each step you select from the table any two disks of different colors and exchange them with two disks of the third color from the extra supply (so there are still 2002 disks on the table). Prove that it is possible after a finite number of steps to have all of the disks on the table of the same color. Moreover, show that the final color is independent of your sequence of steps.
**Solution** : Denote \( r, w \), respectively by, the number of disks of color red, white, respectively blue. At any step \( r + w + b = 2002 \), so two of these numbers are equal mod 3. We can suppose without costs, \( r = b \) mod 3. The possible transformations are described by the new numbers of disks \( \{r + 2, w - 1, b - 1\} \), \( \{r - 1, w + 2, b - 1\} \), and \( \{r - 1, w - 1, b + 2\} \). In all these cases the new number of red disks is equal mod 3 with the new number of blue disks. Then if after a number of steps we will have only one color this must be white, since \( r = b = 0 \) in this case.

Let us prove we can obtain \( r = b = 0 \) after a number of steps. If \( r = b \) then the problem is solved. Suppose \( r > b \). There is a positive integer \( k \) such that \( r = b + 3k \). Making the transforms \( \{r - 1, w + 2, b - 1\} \) and \( \{r - 2, w + 1, b + 1\} \) the difference between the number of red and blue disks will decrease by 3. After \( k \) such pairs of transforms we will have the same number of red and blue disks.

**Problem 2.6.** On the island of Camelot live 13 grey, 15 brown and 17 crimson chameleons. If two chameleons of different colors meet they both simultaneously change color to the third color (e.g. if a grey and a brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be the same color?

**Solution** : Let \( g, b \), respectively \( c \), be the remainder of the number of grey, brown, respectively crimson, chameleons. Then \( \{g, b, c\} = \{0, 1, 2\} \), independently on the color changes. So the answer is negative.

**Problem 2.7.** A machine gives out five pennies for each nickel inserted into it. The machine also gives out five nickels for each penny. Can Peter, who starts out with one penny, use the machine in such a way to end up with an equal number of nickels and pennies?

**Solution** : With each use of the machine Peter gets 4 more coins, so the number of coins he has is of the form \( 4k + 1 \). If Peter will end up with the same number of pennies and nickels, the number of its coins will be even. Contradiction.

**Problem 2.8.** Suppose the positive integer \( n \) is odd. First Al writes the numbers 1, 2, ..., \( 2n \) on the blackboard. He picks any two numbers \( a, b \), erases them and writes instead, \( |a - b| \). Prove that an odd number will remain at the end.

**Problem 2.9.** Suppose not all four numbers \( a, b, c, d \) are equal. Start with \( (a, b, c, d) \) and repeatedly replace \( (a, b, c, d) \) with \( (a - b, b - c, c - d, d - a) \). Then at least one number of the quadruple will become arbitrarily large.

**Problem 2.10.** Each one of the numbers \( a_1, a_2, ..., a_n \) is 1 or \(-1\), and we have \( a_1a_2a_3a_4 + a_2a_3a_4a_5 + ... + a_na_1a_2a_3 = 0 \). Prove that \( n \) is divisible by 4.

**Problem 2.11.** Assume an usual \( 8 \times 8 \) chessboard with the usual coloring. You may repaint all squares of a row or column, or any \( 2 \times 2 \) square. The goal is to attain just one black square. Can you reach the goal?

**Problem 2.12.** Solve the equation \( (x^2 - 3x + 3)^2 - 3(x^2 - 3x + 3) + 3 = x \).

**Problem 2.13.** Does the sequence of perfect squares contain an infinite arithmetic subsequence?
Problem 2.14. Is it possible to transform \( f(x) = x^2 + 4x + 3 \) into \( g(x) = x^2 + 10x + 9 \) by a sequence of transformations of the form \( f(x) \rightarrow x^2f(1/x + 1) \) or \( f(x) \rightarrow (x - 1)^2f[1/(x - 1)] \)?

Problem 2.15. a) For which \( n \times n \) chess board with a corner removed can be tiled by \( 3 \times 1 \) dominoes?

b) For which \( n \times n \) chess board with a square removed can be tiled by \( 3 \times 1 \) dominoes?

2. Pigeonhole principle

Problem 2.16. Let \( n \geq 3 \) be odd and let \( a_1, a_2, \ldots, a_n \) be a rearrangement of the integers \( 1, 2, \ldots, n \). Prove that \( (a_1 - 1)(a_2 - 2) \cdots (a_n - n) \) is even.

Solution: Let \( n = 2k + 1 \). Between the numbers \( 1, 2, \ldots, 2k + 1 \) there are \( k \) even numbers. Then at least one of the \( k + 1 \) numbers \( a_1, a_3, \ldots, a_{2k+1} \), say \( a_{2j+1} \), is odd, and \( a_{2j+1} - (2j + 1) \) is even.

Problem 2.17. Let \( A \) be any set of 19 distinct integers chosen from the arithmetic progression \( 1, 4, 7, 10, \ldots, 100 \). Prove that there must be two distinct integers in \( A \) whose sum is 104.

Solution: The numbers in the arithmetic progression are of the form \( 1 + 3k \) with \( k \) taking values in the set \( 0, 1, 2, \ldots, 33 \). It suffices to show that among 19 numbers chosen in this set there are two with the sum 34. At least 17 numbers are from the set \( 1, 2, 3, \ldots, 16, 18, 19, \ldots, 33 \). Arranging the elements of the set in pairs \( (1, 33), (2, 32), \ldots, (16, 18) \) we see that at least two of the 17 numbers have to be chosen from the same pair, so their sum is 34.

Problem 2.18. Let \( A = \{500, 501, 502, \ldots, 550\} \). Prove that we cannot partition \( A \) into 5 subsets with the sum of elements of each subset strictly less than 5555.

Solution: Suppose \( A = \bigcup_{i=1}^{5} A_i \). Since \( A \) has 51 elements, at least one of the \( A_i \)'s has 11 elements. The sum of these elements is at least \( 500 + 501 + 502 + \ldots + 510 = 5555 \).

Problem 2.19. Let \( S \) be a square of side 2, and choose 9 points inside \( S \). Show that 3 of these points may be chosen which are the vertices of a triangle of area \( \leq 1/2 \).

Solution: We partition the square in 4 squares with the length of the side 1. Then there is a square which contains at least 3 of the points. One can see first that the area of this triangle is smaller than the area of a triangle with the vertices on the sides of the square (obtained for example extending the sides of the original triangle) and then notice that the area of this second triangle is smaller that the area of a triangle with the vertices in the vertices of the square. But the area of such a triangle is 1/2.

Problem 2.20. Let \( S \) be a square of side 1 and 5 points in the interior of \( S \). Show that there are 2 points among these 5 points, such that the distance between them is less than \( \sqrt{2}/2 \).
Solution: Consider the partition of the square in 4 squares of side 1/2. Then one of these squares contain at least two of the points. The distance between these points is smaller than the diagonal $\sqrt{2}/2$ of the square.

Problem 2.21. Consider 28 points inside a cube of side 1. Prove that at least two of them are at a distance $\leq \frac{\sqrt{3}}{3}$.

Problem 2.22. Nineteen points are chosen inside a regular hexagon whose sides have length 1. Prove that two of these points may be chosen whose distance apart is at most $1/\sqrt{3}$.

Solution: Consider the partition of the hexagon in 6 equilateral triangles of side 1. Then in one of these triangles there are 4 points. Let $ABC$ be this triangle, $D, E, F$, respectively, the midpoints of the sides $AB, BC, CA$, and $O$ the center of the circle inscribed in $ABC$. Then in one of the quadrilaterals $ADOF$, $BDOE$, $CEOF$ there are two of the points. The distance between these points is smaller than $1/\sqrt{3}$.

Problem 2.23. In a parliament each member has at most 3 enemies. Prove that the parliament can be divided into two chambers such that each member will be in a room with at most one enemy.

Problem 2.24. For a class with two or more students, show that at least two students have the same number of friends. Assume that you cannot be your own friend. Also assume that if I am your friend, then you are my friend (and vice versa).

Solution: Let $n$ be the number of students. A student can have a number of friends between 0 (if we doesn’t have any friends) and $n-1$ (if all the other students are his friends). If all the students have distinct number of students then one can label them from 1 to $n$ such that the student $k$ has $k-1$ friends. But then the student $n$ is friend with all the other students, including the student 1 which doesn’t have any friends. Contradiction.

Problem 2.25. Show that at any party there are two people who know the same number of other guests.

Solution: Reformulation of the previous exercise.

Problem 2.26. Mr. and Mrs. Smith went to a party attended by 15 other couples. Various handshakes took place during the party. In the end Mrs. Smith asked each person at the party how many handshakes did they have. To her surprise, each person gave a different answer. How many hand shakes did Mr. Smith have? (Here we assume that no person shakes hand with his/her spouse and of course with itself).

Solution: There are 32 people at the party and Mrs. Smith had 31 answers. Each person can have between 0 and 30 handshakes and all the answers were different, so one can label the people at the party (except Mrs. Smith) from 0 to 30, the number being the number of handshakes. The person 30, had a handshake with everybody except himself and his/her spouse, so his/her spouse must be number 0. We take
out this couple, and we consider all the handshakes between the others. One can relabel the people from 0 to 28 (we subtract the handshake that everybody had with number 30). The same argument shows that the person 28 is married with the person 0. We take out this couple, we relabel, and we continue in this way till we see that Mr. Smith is number 15.

**Problem 2.27.** Given a positive integer \( n \), show that there exists a positive integer containing only the digits 0 and 1 in its decimal notation, and which is divisible by \( n \). If \( n \) is not divisible by 2 or 5, then there is a multiple of \( n \) containing only the digit 1 in its decimal notation.

**Solution:** Consider the sequence 1, 11, 111, 1111, ... Between the first \( n+1 \) terms of this sequence there are two numbers \( p \) and \( q \) which are equal mod \( n \). Then the difference \( p - q \) is divisible by \( n \) and is of the form \( N = 11 \ldots 100 \ldots 0 \). If \( n \) is relatively prime with 10, then \( n \) divides the number 11 \ldots 1 obtained from \( N \) taking out the 0's.

**Problem 2.28.** Given a set of \( n+1 \) positive integers, none of which exceeds 2\( n \), show that at least one member of the set must divide another member of the set.

**Solution:** Write all the numbers from 1 to 2\( n \) under the form \( 2^n k (2^m k + 1) \). Since we have \( n+1 \) numbers and the odd part of them can take \( n \) possible values 1, 3, 5, ..., 2\( n \) - 1, two numbers will have the same odd part. The one with the smaller power of 2 will divide the other one.

**Problem 2.29.**

a) Let \( S \) be a set of \( n \) distinct positive integers, none of which exceeds 2\( n \). Prove that one of these numbers is divisible by \( n \) or \( S \) contains two elements with the sum divisible by \( 2n \).

b) Prove that among \( n+1 \) distinct positive integers, none of which exceeds 2\( n \), one of them is \( n \), or the sum of two numbers is 2\( n \).

**Solution:** a) Suppose none of these numbers is divisible by \( n \). Then we have \( n \) numbers from a set that can be partitioned in the reunion of the \( n - 1 \) sets \{1, 2\( n \) - 1\}, \{2, 2\( n \) - 2\}, ..., \{n - 1, n + 1\}. Using the pigeonhole principle we will have in \( S \) both numbers from one of these sets.

b) Consequence of a).

**Problem 2.30.** Let \( S \) be a set of 1003 distinct positive integers, none of which exceeds 2004. Prove that \( S \) contains two elements with the sum divisible by 1002.

**Solution:** Reformulation of the previous problem.

**Problem 2.31.**

a) Let \( S \) be a set of 51 positive integers, none of which exceeds 99. Prove that \( S \) contains two elements with the sum in \( S \).

b) Let \( S \) be a set of 51 positive integers, none of which exceeds 99. Prove that \( S \) contains two elements with the difference in \( S \).

**Solution:** a) The problem is equivalent with proving that \( S \) contains two elements \( x > y \) such that \( x - y \) \( \in \) \( S \). Let \( S = \{x_1, x_2, ..., x_{51}\} \). Since \( T = \{x_{51} - x_1, x_{50} - x_1, ..., x_2 - x_1\} \) are 50 distinct between 1 and 99, \( T \) and \( S \setminus \{x_1\} \) have a common element.
b) Same proof as a).

**Problem 2.32.** Let $S$ be a set of 101 distinct integers, none of which exceeds 99 in absolute value. Prove that $S$ contains 3 numbers with the sum 0.

**Solution:** We distinguish 2 cases. If $0 \in S$, then we have 100 more numbers in $S$. $S$ will then necessarily contain both elements of one the pairs $(-99, 99)$, $(-98, 98)$, $\ldots$, $(-1, 1)$.

If $0 \notin S$

**Problem 2.33.** Let $S$ be a set of seven distinct positive integers which are less than or equal to 24. Prove that $S$ has two subsets whose sums are equal.

**Solution:** Let $1 \leq x_1 < x_2 < \ldots < x_7 \leq 24$ be the seven numbers. Any non-empty subset of $X = \{x_1, x_2, \ldots, x_7\}$ is giving a sum, hence we have $2^7 - 1 = 127$ sums. If we have 4 consecutive integers in $X$, say $a, a + 1, a + 2, a + 3$, then $a + (a + 3) = (a + 1) + (a + 2)$.

Consider now the case when there are no 4 consecutive integers in $X$. Then $x_1 \leq 17, x_2 \leq 18, x_3 \leq 19, x_4 \leq 20, x_5 \leq 22, x_6 \leq 23, x_7 \leq 24$. Indeed, if $x_4 \geq 21$, then necessarily $x_4 = 21, x_5 = 22, x_6 = 23, x_7 = 24$. Contradiction.

The sums of the subsets of $X$ are then integers between $x_1$ and $x_1 + (18 + 19 + 20 + 22 + 23 + 24) = x_1 + 126$. We have 127 sums taking values in a set of 127 elements. If two of them are equal the problem is solved. Supposing they are all distinct, we have necessarily $x_2 = 18, x_3 = 19, x_4 = 20, x_5 = 22, x_6 = 23, x_7 = 24$. But then $x_2 + x_7 = 40 = x_3 + x_6$. Contradiction.

**Problem 2.34.** Let $n$ be a positive integer and $x$ a positive real number. Show there exist integers $a, b$ such that $1 \leq a \leq n - 1$ and $|ax - b| \leq \frac{1}{n}$.

**Solution:** Consider the $n + 1$ fractionary part of the numbers $0, x, 2x, 3x, \ldots, nx$ and the partition of the interval $[0, 1]$ in the $n$ intervals $\left[\frac{k}{n}, \frac{k + 1}{n}\right)$, where $k = 0, 1, \ldots, n - 1$. Then, there are integers $k \in \{0, 1, 2, \ldots, n - 1\}$ and also $p, q \in \{1, 2, \ldots, n\}$ such that $px = m + \alpha, qx = n + \beta$, with $m, n$ positive integers and $\alpha, \beta \in \left[\frac{k}{n}, \frac{k + 1}{n}\right]$. Taking $a = p - q$ and $b = m - n$ we have $|ax - b| = |\alpha - \beta| \leq \frac{1}{n}$.

**Problem 2.35.** Each square of a $3 \times 7$ chessboard is colored either black or white. Show that there is a rectangle with sides parallel to the edges of the board all of whose corners are squares of the same color. For a $3 \times 6$ chessboard the conclusion remains true?

**Solution:** Let $A$ and $B$ be the two colors. If two rows have the same distribution of colors, then the problem is solved. Suppose all the rows have distinct distributions of colors. Since we have 8 possible distributions with two of them, AAA and BBB having all the squares of the same color and 7 rows, one row has all the squares of the same color, say AAA. But among the 7 rows we have at least one of AAB and ABA, and the problem is solved.

The $3 \times 6$ chessboard with the rows $AAB, ABA, BAA, ABB, BAB, BBA$ is a counterexample for the second part.
Problem 2.36. Prove that every polynomial over the complex numbers has a nonzero polynomial multiple whose exponents are all divisible by 100.

Solution: Let $P(x)$ be a polynomial of degree $n$. For any $k = 1, 2, \ldots, n - 1, n$, there are polynomials $Q_k, R_k$, with $\deg R_k \leq n - 1$, such that

$$x^{100k} = Q_k(x)P(x) + R_k(x)$$

The $n$ polynomials $R_1(x), R_2(x), \ldots, R_n(x)$ are linearly dependent in the vector space of polynomials of degree smaller than $n - 1$. Then there are complex numbers $a_1, a_2, \ldots, a_n$, not all zero, such that $a_1R_1(x) + a_2R_2(x) + \ldots + a_nR_n(x) = 0$. Consequently,

$$a_1x^{100} + a_2x^{200} + \ldots + a_nx^{100n} = (a_1Q_1(x) + a_2Q_2(x) + \ldots + a_nQ_n(x))P(x)$$

Problem 2.37. A is a subset of a finite multiplicative group $G$, and $A$ contains more than one half of the elements of $G$. Prove that each element is a product of two elements of $A$.

Solution: Let $g$ be an element of $G$. The function $f: A \rightarrow G$, $f(a) = a^{-1}g$ is one-to-one, so the number of elements of $f(A)$ is the number of elements of $A$. But then $A$ and $f(A)$ overlap, i.e. there are $a, b \in A$ such that $f(a) = b$, which corresponds to $g = ab$.

Problem 2.38. Prove there exist integers $a, b, c$, not all 0 and each of absolute value less than $10^6$ such that $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$.

Solution: Denote $A$ the set of integers of absolute value less than $5 \cdot 10^5$. Each of the integers $a, b$ and $c$ can take $2(10^6 - 1) - 1$ values. Since $a + b\sqrt{2} + c\sqrt{3}$ has distinct values for different triples $(a, b, c)$, then $10^{11}(a + b\sqrt{2} + c\sqrt{3})$ can take $(2 \cdot 10^6 - 1)^3 > 7 \cdot 10^{18}$ distinct values. But $|10^{11}(a + b\sqrt{2} + c\sqrt{3})| \leq 10^{11}(1 + \sqrt{2} + \sqrt{3})(10^6 - 1) < 5 \cdot 10^{17}$, so there are $a_1, a_2, b_1, b_2, c_1, c_2$ of absolute values less than $10^6$ such that $10^{11}(a_1 + b_1\sqrt{2} + c_1\sqrt{3})$ and $10^{11}(a_2 + b_2\sqrt{2} + c_2\sqrt{3})$ have the same integer part. Consequently, $|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}$, for $a = a_1 - a_2$, $b = b_1 - b_2$, and $c = c_1 - c_2$.

Problem 2.39. Let $S$ be a set consisting of $k = 10$ different integers between 1 and $n$ (inclusive).

a) If $n = 105$, prove there exist two disjoint nonempty subsets of $S$ whose elements have the same sum.

b) If $n = 110$, prove there exist two disjoint nonempty subsets of $S$ whose elements have the same sum.

c) Solve the problem with $k = 5$ and $n = 11$.

d) For a given $k$ find the highest value of $n$ for which the statement holds.

Solution: a) With the integers $x_1, x_2, \ldots, x_{10}$ we can make $2^{10} - 1 = 1023$ sums. The highest value these sums can get is $105 + 104 + 103 + \ldots + 96 = 1005$. Then necessarily there are two equal sums. Reducing, if necessary, the common terms we obtain sums of disjoint nonempty subsets.

b) The proof is similar, only we have to see that in the subset $\{101, 102, \ldots, 110\}$ we already have $107 + 110 = 109 + 108$, etc.
Problem 2.40. Show that between any 9 (respectively 5) different lattice points in the space (respectively in plane) one can choose two such that the line segment joining them has a lattice point in its interior.

Solution: There are $2 \cdot 2 \cdot 2 = 8$ possible triples of parities for any point $(x,y,z)$ of integer coordinates. Then among the 9 points there are two with exactly the same parity for each of the coordinates and as a consequence the midpoint of the segment determined by them has integer coordinates.

Problem 2.41. Let $M$ be a set of $n$ integers (not necessarily distinct). Prove that $M$ has a subset with the sum of elements divisible by $n$.

Problem 2.42. Consider a sequence consisting of $n^2 + 1$ distinct integers. Prove that this sequence has a monotone (increasing or decreasing) subsequence of length $n + 1$.

Problem 2.43. If $A$ is a finite set and $f : A \to A$ is a function, prove there are $p, q$ positive integers such that $f^{(n)} = f^{(n+p)}$ for any $n$ integer, $n \geq q$. We denoted $f^{(n)} = f \circ f \circ \ldots \circ f$, where the $f$ appears $n$ times in the composition.

Solution: Since $A$ is finite there is a finite number of functions from $A$ to $A$. Then there are $q < m$ positive integers such that $f^{(m)} = f^{(q)}$. Indeed in the contrary case there will be infinitely many functions from $A$ to $A$. Taking $p = m - q$ we have $f^{(q)} = f^{(q+p)}$ and we compose by $f^{(n-q)}$.

Problem 2.44. Let $k$ be a positive integer. Prove that among $k + 2$ integers there are two with the sum or the difference divisible by $2k + 1$.

Solution: If two of the numbers are equal mod $2k + 1$, then their difference is divisible by $2k + 1$. If all the $k + 2$ numbers have distinct remainders mod $2k + 1$, then there are at least $k + 1$ distinct non-zero remainders mod $2k + 1$. Let us denote these remainders $1 \leq r_1 < r_2 < \ldots < r_{k+1} \leq 2k$. We have then $1 \leq (2k+1) - r_{k+1} < (2k+1) - r_k < \ldots < (2k+1) - r_1 \leq 2k$. The subsets \{r_1, r_2, \ldots, r_{k+1}\} and \{(2k+1) - r_{k+1}, \ldots, (2k+1) - r_1\} have the positive integers equal at most to $2k$ have each $k + 1$ elements and therefore cannot be disjoint. There are $r_i \neq r_j$ such that $r_i = (2k+1) - r_j$.

Problem 2.45. Prove that if six people are riding together in an elevator, there is either a three-person subset of mutual friends (each knows the other two) or a three-person subset of mutual strangers (each knows neither of the other two).

Solution: Let $A$ be one of these 6 people. Then between the remaining 5 he either is friend with at least 3 of them, or he doesn’t know at least 3 of them.

Suppose $A$ has 3 friends, say $B$, $C$, and $D$. Then if among these 3 people there are two mutual friends, say $B$ and $C$, then $A$, $B$ and $C$ are mutual friends. If among $B$, $C$, and $D$ there are no two mutual friends, then $B$, $C$ and $D$ are mutual strangers.

Suppose $A$ doesn’t know any of $B$, $C$ and $D$. Then if among these 3 people there are two mutual strangers, say $B$ and $C$, then $A$, $B$ and $C$ are mutual strangers. Otherwise, $B$, $C$ and $D$ are mutual friends.
3. The Principle of Induction

Consider a mathematical statement $S(n)$ which depends on the integer $n$. We have the following types of induction:

**Type I:** We verify $S(n_0)$, and we prove that $S(n + 1)$ is a consequence of $S(n)$. Then the statement $S(n)$ is valid for all the values of $n \geq n_0$.

**Example 2.1.** All the men have the same height.

**Problem 2.46.** Prove with and without induction that for any integer $n \geq 1$ we have

$$(n + 1)(n + 2) \ldots (2n) = 2^n \cdot 1 \cdot 3 \cdot \ldots \cdot (2n - 1)$$

**Type II:** We verify $S(n_0), S(n_0 + 1), \ldots, S(n_0 + k - 1)$ and we prove that $S(n + k)$ is a consequence of $S(n)$. Then the statement $S(n)$ is valid for all the values $n \geq n_0$.

**Problem 2.47.** Determine the positive integers $n$ such that any square can be divided into $n$ squares.

**Problem 2.48.** Prove that any integer greater than 8 is of the form $3p + 5q$, with $p, q$ non-negative integers.

**Type III:** We verify $S(n_0)$ and we prove that $S(n + 1)$ is a consequence of $S(n_0), S(n_0 + 1), \ldots, S(n - 1), S(n)$. Then the statement $S(n)$ is valid for all the values $n \geq n_0$.

**Problem 2.49.**

a) Let $x$ be a complex number such that $x + 1/x \in \mathbb{Z}$. Prove that $x^n + 1/x^n \in \mathbb{Z}$.

b) Let $z$ be a complex number such that $z + 1/z$ is a real number of absolute value $\leq 2$. Prove that $z^n + 1/z^n$ is a real number of absolute value $\leq 2$.

**Problem 2.50.** Let $x$ be a real number such that $\sin x + \cos x$ is rational. Prove that $\sin^n x + \cos^n x$ is a rational number for any integer $n$. We will use induction observing that

$$\sin^{n+1} x + \cos^{n+1} x = (\sin^n x + \cos^n x)(\sin x + \cos x) - \sin x \cos x(\sin^{n-1} x + \cos^{n-1} x)$$

**Solution:** Using $\sin^2 x + \cos^2 x = 1$, we obtain that

$$\sin x \cos x = \frac{(\sin x + \cos x)^2 - \sin^2 x - \cos^2 x}{2}$$

is a rational number.

**Type IV:** We verify $S(n_0)$ and the we prove that $S(2n)$ and $S(n - 1)$ are consequences of $S(n)$. Then the statement $S(n)$ is valid for all the values $n \geq n_0$.

**Problem 2.51.** Let $x_1, x_2, \ldots, x_n \geq 1$. Prove that

$$(1 + \sqrt{x_1 x_2 \ldots x_n}) \left( \frac{1}{1 + x_1} + \frac{1}{1 + x_2} + \ldots + \frac{1}{1 + x_n} \right) \geq n$$

**Solution:** With $f(x) = \frac{1}{1 + x}$ the inequality to prove becomes

$$S_n : \frac{f(x_1) + f(x_2) + \ldots + f(x_n)}{n} \geq f(\sqrt[n]{x_1 x_2 \ldots x_n})$$
We prove it by induction.
For \( n = 1 \) the inequality is an obvious equality. For \( n = 2 \) the inequality is equivalent to

\[
(\sqrt{x_1x_2} - 1)(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0
\]

Assuming \( S_n \) true and using also \( S_2 \) we prove \( S_{2n} \) true. Then assuming \( S_{n+1} \) we prove \( S_n \). By the induction principle, \( S_n \) is then true for all \( n \in \mathbb{N} \).

**Problem 2.52.** Let \( a_1, a_2, \ldots, a_n \) and \( b_1, b_2, \ldots, b_n \) be nonnegative real numbers. Show that

\[
(a_1a_2\cdots a_n)^{1/n} + (b_1b_2\cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2)\cdots (a_n + b_n)]^{1/n}.
\]

**Type V: Inverse induction**

**Problem 2.53.** For any natural \( n \) prove the inequality

\[
\sqrt[1]{2} \sqrt[3]{3} \sqrt[4]{4} \cdots \sqrt[n]{n} < 3
\]

**Hint:** Use inverse induction after \( m \) to prove

\[
\sqrt[m]{m+1} \sqrt[m+2]{m+2} \cdots \sqrt[n-1]{n} \sqrt[n]{n} < m + 1
\]

or use the inequality between the arithmetic and geometric means.

**Problem 2.54.** Prove with and without induction that for any integer \( n \geq 2 \) we have

\[
\sqrt{n} < 2(\sqrt{n + 1} - 1) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}
\]

**Problem 2.55.** Determine the integers \( x \) such that \( 3x + 1 < 2 \log_2(x + 4) \).

**Problem 2.56.** *(Fermat’s theorem)* Let \( p \) be a prime number. Then for any integer \( n \), \( n^p - n \) is divisible by \( p \).

**Problem 2.57.** Prove that for any integer \( n \), \( n^5 - n \) is divisible by 30.

**Problem 2.58.** Prove that for any positive integer \( n \) and any real number \( x \), we have \( |\sin nx| \leq n|\sin x| \).

**Problem 2.59.** The Fibonacci sequence is defined through \( f_0 = 0, f_1 = 1, \) and \( f_{n+2} = f_{n+1} + f_n \). Prove that

a) \( f_1 + f_2 + \ldots + f_n = f_{n+2} - 1 \)
b) \( f_{n-1}f_{n+1} = f_n^2 + (-1)^n \).
c) \( f_1 + f_3 + \ldots + f_{2n+1} = f_{2n+2} \)
d) \( \sum_{k=1}^{n} f_k^2 = f_nf_{n+1} \)

**Problem 2.60.** Prove there are infinitely many positive integers \( n \) such that \( n \) is a divisor of \( 2^n + 1 \).
Problem 2.61. For what positive integers \( n \) is \( x^n + 1/x^n \) expressible as a polynomial with real coefficients in \( x - 1/x \)?

Problem 2.62. All numbers of the form 1007, 10017, 100117, ... are divisible by 53.

Problem 2.63. Prove with and without induction that for any integer \( n \geq 1 \) we have
\[
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + ... + \frac{1}{2n}.
\]

Problem 2.64. Prove that a set with \( n \) elements has \( 2^n \) subsets.

Problem 2.65. Let \( z \) be a complex number and \( n \geq 2 \) an integer. Prove there are real numbers \( a \) and \( b \) such that \( z^n = az + b \).

Problem 2.66. Determine the integers \( n \) such that \( 2^n > n^2 \).

Problem 2.67. Let \( n \geq 1 \) be an integer. Prove that \((n+1)(n+2)(n+3)...(n+n)\) is divisible by \( 2^n \) but not by \( 2^{n+1} \).

Problem 2.68. For \( n \geq 1 \) integer let \( P_n = \prod_{k=1}^{2^n-1} (2k-1) \). Prove that \( P_n - 1 \) is a multiple of \( 2^n \) for any \( n \geq 3 \). If we subtract 1 from the product of the first \( 2^{n-1} \) odd numbers we obtain a multiple of \( 2^n \).

Solution: We proceed by induction. We have \( P_3 = 1 \cdot 3 \cdot 5 \cdot 7 = 105 \), therefore \( P_3 - 1 = 104 \) is divisible by \( 2^3 \). Assume \( P_n - 1 \) is divisible by \( 2^n \). Then \( P_{n+1} = P_n \cdot (2^n + 1)(2^n + 3)...(2^{n+1} - 1) = P_n[M2^n + (1 \cdot 3 \cdot ... \cdot (2^n - 1))] = MP_n2^n + P_n^2 \),
so \( P_{n+1} - 1 = MP_n2^n + (P_n - 1)(P_n + 1) \) is a multiple of \( 2^n \cdot 2 = 2^{n+1} \).
CHAPTER 3

Set theory

PROBLEM 3.1. Let $X$ be a set. For any set $A \subset X$, the characteristic function of $A$ is defined by $\chi_A : X \to \{0, 1\}$, $\chi_A(x) = 1$ if $x \in A$, and $\chi_A(x) = 0$ if $x \notin A$. Prove that:

1. $\chi_\emptyset = 0$, $\chi_X = 1$.
2. $\chi_A = \chi_B$ if and only if $A = B$.
3. $\chi_{A \cap B} = \chi_A \chi_B$.
4. $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \chi_B$.
5. $\chi_A^c = 1 - \chi_A$.
6. $\chi_{A \setminus B} = \chi_A (1 - \chi_B)$.

Solution: Straightforward.

PROBLEM 3.2. Prove that $(A \Delta B) \Delta C = A \Delta (B \Delta C)$.

Solution: Straightforward with the characteristic function.

PROBLEM 3.3. Let $(A_n)_{n\in \mathbb{N}}$ be a sequence of sets. Define $\lim_{n \to \infty} A_n = \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k$ and $\lim_{n \to \infty} A_n = \bigcup_{n \geq 1} \bigcap_{k \geq n} A_k$. The sequence $(A_n)$ is called convergent is $\lim_{n \to \infty} A_n = \lim_{n \to \infty} A_n$.

i) Give an example of a sequence of sets such that $\lim_{n \to \infty} A_n \neq \lim_{n \to \infty} A_n$.

ii) Prove that $(\lim_{n \to \infty} A_n)^c = \lim_{n \to \infty} A_n^c$.

iii) $\chi_{\lim_{n \to \infty} A_n} = \lim_{n \to \infty} \chi_{A_n}$.

iv) A sequence of sets is convergent if and only if the sequence of the associated characteristic functions is convergent.

v) Let $(E_n)$ be a sequence of sets and define $D_1 = E_1$, $D_n = D_{n-1} \Delta E_n$. Then $(D_n)$ is convergent if and only if $\lim_{n \to \infty} E_n = \emptyset$.

Solution: (i) Let $A \neq B$ be two sets and define $A_{2n} = A$, $A_{2n+1} = B$. Then $\lim_{n \to \infty} A_n = A \cap B \neq A \cup B = \lim_{n \to \infty} A_n$.

(ii), (iii), (iv) Straightforward.

PROBLEM 3.4. $(\mathcal{P}(\Omega), \Delta, \bigcap)$ is a ring isomorphic with $(\mathbb{Z}_2^\Omega, +, \cdot)$.

Solution: The isomorphism is the characteristic function.

PROBLEM 3.5. Given 10 distinct sets, how many sets we obtain using the intersection, the reunion, and the difference of sets?
**Solution**: Given \( n \) sets \( A_1, A_2, \ldots, A_n \) we can obtain at least \( n \) and at most \( 2^n - 1 \) distinct and disjoint sets, through intersection and difference of sets. On the other side, given \( k \) disjoint sets, we get \( 2^k \) possible different reunions. Therefore, the answer of the problem is \( 2^k \) with \( 10 \leq k \leq 2^{10} - 1 \).

**Problem 3.6.** Let \( R \) be the region consisting of the points \((x, y)\) of the cartesian plane satisfying both \(|x| - |y| \leq 1\) and \(|y| \leq 1\). Sketch the region \( R \) and find its area. [P1988]

**Solution**: The region is the rectangle \((-2, 1), (2, 1), (2, -1), (-2, -1)\) less the two triangles \((2, 1), (1, 0), (2, -1)\) and \((-2, 1), (-1, 0), (-2, -1)\). Hence the area is \(8 - 2.1 = 6\).

**Problem 3.7.** Let \( A \) be a set with \( n \) elements and \( A_1, A_2, \ldots, A_p \) subsets of \( A \) each having \( n - 2 \) elements. If \( A_i \cup A_j \cup A_k \neq A \) for any \( i, j, k \), prove that \( \bigcup_{i=1}^{p} A_i \neq A \).

**Solution**: Suppose \( \bigcup_{i=1}^{p} A_i = A \). Then the two elements of \( A \setminus A_1 \) are in \( \bigcup_{i=2}^{p} A_i \). There are \( j, k \) such that \( A \setminus A_1 \subset A_j \cup A_k \), so \( A = A_1 \cup A_j \cup A_k \). Contradiction.
CHAPTER 4

Binomial theorem and coefficients

**Problem 4.1.** Determine \( \sum_{k=0}^{n} \binom{2k}{3n} \).

**Problem 4.2.** Prove that the binomial coefficient \( \binom{n}{k} \) is a multiple of \( k \) for all \( k, n \in \mathbb{N} \).

**Solution:** Use the identity \( \binom{n}{k} = k \binom{n-1}{k-1} \).

**Problem 4.3.** Let \( n > k \) be positive integers.

a) Prove that \( \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} \).

b) Evaluate \( \sum_{i=3}^{n} \binom{i}{3} \).

**Problem 4.4.** a) Prove that for any real number \( x \) and positive integer \( n \) we have

\[ n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} x^{k-1} \]

b) Evaluate the sum \( \sum_{k=0}^{n} (k+1) \binom{n}{k} \).
1. General properties of functions

Problem 5.1. If $f$ is a real valued function such that $f(f(x)) = 2x - 1$, for all $x \in \mathbb{R}$, determine $f(1)$.

Solution: Using the associativity of the composition of functions we have

$$2f(x) - 1 = [(f \circ f) \circ f](x) = [f \circ (f \circ f)](x) = f(2x - 1), \forall x \in \mathbb{R}$$

Taking $x = 1$ we get $2f(1) - 1 = f(2 \cdot 1 - 1)$, so $f(1) = 1$.

Problem 5.2. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = 0$ if $x < 1$ and $f(x) = 2x - 2$ if $x \geq 1$. Determine the solutions of the equation $f(f(f(f(x)))) = x$.

Solution: We determine in two steps the expression of the composition $f \circ f \circ f \circ f$.

We have

$$g(x) = f(f(x)) = \begin{cases} 0, & f(x) < 1 \\ 2f(x) - 2, & f(x) \geq 1 \end{cases} = \begin{cases} 0, & x < \frac{3}{2} \\ 4x - 6, & x \geq \frac{3}{2} \end{cases}$$

and from here

$$f(f(f(f(x)))) = g(g(x)) = \begin{cases} 0, & x < \frac{15}{8} \\ 16x - 30, & x \geq \frac{15}{8} \end{cases}$$

We can now find easily that the all solutions of the equation $f(f(f(f(x)))) = x$ are $x = 0$ and $x = 2$.

Second solution: For an increasing function $f$ from $f(f(x)) = x$ it follows that $f(x) = x$.

Problem 5.3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = 0$ if $x < 1$ and $f(x) = 2x - 2$ if $x \geq 1$. For $n$ positive integer determine $(f \circ f \circ \ldots f)(x)$.

Solution: We prove by induction that

$$(f \circ f \circ \ldots f)(x) = \begin{cases} 0, & x < \frac{2^n - 1}{2^{n-1}} \\ 2^n x - 2^{n+1} + 2, & x \geq \frac{2^n - 1}{2^{n-1}} \end{cases}$$
2. Injective, surjective, bijective functions

Problem 5.4. If a function \( f \) is bijective, then the graphs of \( f \) and \( f^{-1} \) are symmetric with respect to the line \( y = x \).

Problem 5.5. Do the graphs of \( f \) and \( f^{-1} \) intersect only on the line \( y = x \)?

Solution: The answer is negative as shown by the function \( f(x) = 1 - x \). We have \( f^{-1} = 1 - x \) and the graphs coincide. FINISH

3. The graph of a function

Problem 5.6. Prove that any function real function \( f : \mathbb{R} \rightarrow \mathbb{R} \) can be written as the sum of two functions whose graphs admit a center of symmetry.

4. Monotonic functions

Problem 5.7. Let \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) be two increasing functions. Prove that \( \max(f, g) \) and \( \min(f, g) \) are also increasing functions.

Solution: Let \( x < y \). Without loss of generality we can assume that \( f(x) \leq g(x) \). Then
\[
\max(f, g)(x) = f(x) \leq f(y) \leq \max(f, g)(y)
\]
Similarly we can assume without any loss of generality that \( f(y) \leq g(y) \). Then
\[
\min(f, g)(y) = g(y) \geq g(x) \geq \min(f, g)(x)
\]

Problem 5.8. Prove that the function \( f : (0, \infty) \rightarrow \mathbb{R}, \ f(x) = x^{x+\frac{1}{2}} \) is increasing.

Solution: We examine the cases:
If \( a > b > 1 \), then \( a + \frac{1}{a} > b + \frac{1}{b} > 0 \), so \( f(a) > a^{b+\frac{1}{2}} > f(b) \).
If \( 1 > a > b > 0 \), then \( 0 < a + \frac{1}{a} < b + \frac{1}{b} \) so \( f(a) > a^{b+\frac{1}{2}} > f(b) \).
If \( a > 1 > b \), then \( f(a) > 1 > f(b) \).

Problem 5.9. Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be an increasing bijection. Then \( f(x) = f^{-1}(x) \) if and only if \( f(x) = x \).

Solution: Obviously if \( f(x) = x \) then \( f(f(x)) = f(x) = x \), so \( f(x) = f^{-1}(x) \). Reciprocally assume that \( f(f(x)) = x \). If \( f(x) > x \) then since \( f \) is increasing we have \( x = f(f(x)) > f(x) \), contradiction. If \( f(x) < x \) then \( x = f(f(x)) < f(x) \), contradiction. Hence \( f(x) = x \).

Problem 5.10. Solve the equation \( e^x - e^{-x} = 2 \ln(x + \sqrt{1+x^2}) \).

Solution: Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be \( f(x) = e^x - e^{-x} \). Then \( f \) is bijective and we observe that \( f^{-1}(x) = \ln(x + \sqrt{1+x^2}) \). Moreover \( f \) is increasing, so according to the previous problem the equation is equivalent with \( f(x) = x \), or \( e^x - e^{-x} = x \).
Consider the function \( g(x) = e^x - e^{-x} - x \). Since
\[
g'(x) = e^x + e^{-x} - 1 = (e^x - e^{-x})^2 + 1 > 0
\]
the function \( g \) is strictly increasing and consequently one to one. We observe that \( g(0) = 0 \) and we conclude that \( x = 0 \) is the only solution of the equation.
CHAPTER 6

Real numbers

1. The decimal representation of real numbers

Problem 6.1. Prove that the number \( \sum_{n=0}^{\infty} 10^{-n^2} \) is irrational.

Solution: The series is convergent to a real number \( a \). Assume \( a \) is rational. Then in its decimal expansion, \( a \) has a period with, say, \( k > 0 \) digits. But for \( n \) big enough, \( 10^{-n^2} \) has more than \( k \) consecutive zeros in the decimal expansion.

Problem 6.2. For \( n \in \mathbb{N}^* \) find the first decimal of \( \sqrt{n^2+n} \).

Solution: Giving a few values to \( n \) we see that
\[
\sqrt{1^2+1} = \sqrt{2} = 1.41\ldots, \quad \sqrt{2^2+2} = \sqrt{6} = 2.44\ldots, \quad \sqrt{3^2+3} = \sqrt{12} = 3.46\ldots
\]
We will prove therefore that the first decimal is 4, which corresponds to the inequalities
\[
n + 0.4 < \sqrt{n^2+n} < n + 0.5 \iff n^2 + \frac{4n}{5} + \frac{4}{25} < n^2 + n < n^2 + n + \frac{1}{4}
\]

Problem 6.3. Prove that any irrational number has at least two digits which appear infinitely many times in its decimal representation. Is there an irrational number with exactly two digits repeating infinitely many times in its decimal representation?

Solution: Suppose there is an irrational number with only one digit appearing infinitely many times in the decimal representation. But then this digit is a period, since the other digits will appear only finitely many times. The number 0.12122122122212... is irrational and has only the digits 1 and 2 in the decimal representation.

Problem 6.4. Consider the Champernowne constant
\[
a = 0.123456789101112131415\ldots
\]
which has the sequence of all positive integers as decimals. Prove that \( a \) is irrational.

Solution: Suppose \( a \) is rational. Then the sequence of decimals in \( a \) has a period \( N \). Let \( k \) be the number of digits in \( N \). We note that \( N \) contains all the digits from 0 to 9, since \( a \) contains infinitely many times each digit. But for any positive integer \( p \), in \( a \) we have the sequence of consecutive digits 11...1, with 1 appearing \( pk \) times, which shows \( N \) is not a period.
2. Rationals and irrationals

Problem 6.6. If \( n \in \mathbb{N} \) is not a perfect square, then \( \sqrt{n} \in \mathbb{R} \setminus \mathbb{Q} \).

Solution: Assume \( \sqrt{n} = \frac{p}{q} \) is in lowest terms (the greatest common divisor of \( p \) and \( q \) is 1). Taking the square we have \( nq^2 = p^2 \) and any prime divisor of \( q \) has to be also a prime divisor of \( p \). It follows that \( q \) doesn’t have any prime divisor, which means \( q = 1 \). Thus \( n = p^2 \). Contradiction.

Problem 6.7. \( \sqrt{6} \) is irrational.

Solution: 6 is not a perfect square, so we use the previous result.

Problem 6.8. Let \( n \geq 1 \) and \( k \geq 2 \) be integers. Prove that \( \sqrt[3]{n} + \sqrt[k]{n} \) and \( \sqrt[3]{n} + \sqrt[4]{n} + \ldots + \sqrt[k]{n} \) are irrational numbers.

Solution: Assume that \( \sqrt[3]{n} + \sqrt[k]{n} = m \) is the rational number. Then \( n + \sqrt[3]{n} = n^k \iff \sqrt[3]{n} = m^k - n \)

It follows that \( \sqrt[3]{n} \) is an integer, say \( p \). From here, \( n = p^k \) and by substitution \( p^k + p = m^k \). But in this case, \( p < m < p + 1 \), contradiction.

If \( \sqrt[3]{n} + \sqrt[4]{n} + \ldots + \sqrt[k]{n} \) is rational, then \( \sqrt[3]{n} + \sqrt[4]{n} \) is rational and we use the first part with \( k = 2 \).

Problem 6.9. If \( a, b \) are positive rational numbers, and \( \sqrt{a} \) is irrational, then \( \sqrt{a} + \sqrt{b} \) is also irrational.

Solution: Assume that \( \sqrt{a} + \sqrt{b} = r \) is rational. Taking the square in \( \sqrt{b} = r - \sqrt{a} \) and arranging, we obtain \( \sqrt{a} = \frac{r^2 + a - b}{2r} \in \mathbb{Q} \), contradiction.

Problem 6.10. Let \( a, b, c \in \mathbb{Q} \) such that \( \sqrt{a} + \sqrt{b} + \sqrt{c} \in \mathbb{Q} \). Prove that \( \sqrt{a}, \sqrt{b}, \sqrt{c} \in \mathbb{Q} \).

Problem 6.11. If \( a_1, a_2, \ldots, a_n \in \mathbb{Q}^{+} \) and \( \sqrt{a_1} \in \mathbb{R} \setminus \mathbb{Q} \), then 
\[ \sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_n} \in \mathbb{R} \setminus \mathbb{Q} \]

Problem 6.12. a) Prove that \( \sqrt{2} + \sqrt{3} \) is irrational.

b) Prove that \( \sqrt{2} + \sqrt{3} + \sqrt{5} \) is irrational.

c) Prove that \( \sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{7} \) is irrational

Solution:

a) First solution: Since 2 and 3 are rationals and \( \sqrt{2} \) is irrational, using the technique of problem 6.9 the sum \( \sqrt{2} + \sqrt{3} \) is irrational.

Second solution: Assume \( \sqrt{2} + \sqrt{3} = r \in \mathbb{Q} \). Taking the square and arranging we obtain \( \sqrt{6} = \frac{r^2 - 5}{2} \in \mathbb{Q} \). Contradiction.
Third solution: We observe that \( a = \sqrt{2} + \sqrt{3} \) is a root of the monic polynomial \( P(x) = \prod (x \pm \sqrt{2} \pm \sqrt{3}) = x^4 - 10^2 + 1 \). If we assume that \( a \) is rational it follows that \( a \) is integer and a divisor of 1. As a positive number it needs to be 1. But \( \sqrt{2} + \sqrt{3} > 1 + 1 = 2 \), contradiction.

b) Similar to second solution of a): Assume \( \sqrt{2} + \sqrt{3} + \sqrt{5} = r \in \mathbb{Q} \).

Then from \((\sqrt{2} + \sqrt{3})^2 = (r - \sqrt{5})^2 \) we obtain \( \sqrt{6} + r \sqrt{5} = r^2 \) and squaring again we get \( \sqrt{30} = \frac{r^4 - 20r^2 - 24}{8r} \in \mathbb{Q} \). Contradiction.

Similar to third solution of a). Consider the monic polynomial \( P(x) = \prod (x \pm \sqrt{2} \pm \sqrt{3} \pm \sqrt{5}) \) of degree 8. We observe that \( a = \sqrt{2} + \sqrt{3} + \sqrt{5} \) is a zero of \( P \). If we assume \( a \in \mathbb{Q} \), then \( a \) is necessarily integer. But \( 1.4 < \sqrt{2} < 1.5 \), \( 1.7 < \sqrt{3} < 1.8 \) and \( 2.2 < \sqrt{5} < 2.3 \) and by addition \( 4.3 < a < 4.6 \). Contradiction.

c) Similar to third solution of a). Consider the monic polynomial of degree 16, \( P(x) = \prod (x \pm \sqrt{2} \pm \sqrt{3} \pm \sqrt{5} \pm \sqrt{7}) \). If \( a = \sqrt{2} + \sqrt{3} + \sqrt{5} + \sqrt{7} \) is rational, then it should be rational as a zero of \( P \). But \( 1.4 < \sqrt{2} < 1.5 \), \( 1.7 < \sqrt{3} < 1.8 \), \( 2.2 < \sqrt{5} < 2.3 \), and \( 2.64 < \sqrt{7} < 2.65 \) and by addition \( 8.01 < a < 8.05 \). Contradiction.

Problem 6.13. Suppose that \( a + \frac{1}{a} \in \mathbb{Q} \). Prove that \( a^n + \frac{1}{a^n} \in \mathbb{Q} \) for all integer \( n > 0 \).

Solution: We proceed by induction. The statement is true for \( n = 1 \). Suppose that \( a^n + \frac{1}{a^n} \in \mathbb{Q} \) for any \( n \leq k \). Then \( a^{k+1} + \frac{1}{a^{k+1}} = \left( a^k + \frac{1}{a^k} \right) \left( a + \frac{1}{a} \right) - \left( a^{k-1} + \frac{1}{a^{k-1}} \right) \in \mathbb{Q} \).

Problem 6.14. Show that between any two real numbers, \( a < b \), there is a rational number \( c \) and an irrational number \( d \).

Solution: There is a positive integer \( n \) such that \( a + \frac{1}{n} < b \), e.g. \( n = \left\lceil \frac{1}{b-a} \right\rceil + 1 \).

Also there is an integer \( m \) such that \( a \leq \frac{m}{n} < a + \frac{1}{n} \), e.g. \( m = \lfloor na \rfloor + 1 \). Then \( c = \frac{m}{n} \) is between \( a \) and \( b \).

Using the first part, there exist a rational number \( r \) such that \( a < r < b \), and there exist a rational number \( s \) such that \( r < s < b \). The number \( d = r + \frac{s-r}{\sqrt{2}} \) is irrational and it lies in the interval \((r,s) \subset (a,b)\).

Problem 6.15. Prove or disprove that an irrational power of an irrational number is irrational.

Solution: The numbers \( a = \sqrt{2} \) and \( b = \log_2 9 \) are both irrational and \( a^b = 3 \).
Second solution. We will prove there exists irrational numbers \( a \) and \( b \) such that \( a^b \) is rational. If the number \( \sqrt{2}^{\sqrt{2}} \) is rational, the problem is solved. Otherwise, consider \( a = \sqrt{2}^{\sqrt{2}} \) and \( b = \sqrt{2} \) and we have \( a^b = 2 \).

**Problem 6.16.** Prove that \( \cos \frac{\pi}{5} \) and \( \cos \frac{\pi}{7} \) are irrational numbers.

**Problem 6.17.** Prove that for any integer \( n \geq 3 \), \( \cos \frac{\pi}{n} \) is an algebraic irrational number.

**Solution:** Use the Chebyshev or Bernoulli polynomials.

3. Radicals

**Problem 6.18.** Nested radicals formula: Let \( a, b \) be integers such that \( a^2 \geq b \geq 0 \). Then

\[
\sqrt{a + \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}
\]

Unnest the nested radicals \( \sqrt{3 + \sqrt{5}} \) and \( \sqrt{4 + \sqrt{7}} \).

**Problem 6.19.** Simplify \( \sqrt{49 + 20\sqrt{6}} + \sqrt{49 - 20\sqrt{6}} \).

**Problem 6.20.** Prove that \( \sqrt{20 + 14\sqrt{2}} + \sqrt{20 - 14\sqrt{2}} = 4 \).

**Solution:** Let \( a = \sqrt{20 + 14\sqrt{2}} \), \( b = \sqrt{20 - 14\sqrt{2}} \), and \( x = a + b \). The equation \( x^3 = a^3 + b^3 + 3abx = 40 + 6x \) has the only real solution \( x = 4 \).

**Second solution** We know \( ab = 2 \) and we have to prove \( a + b = 4 \). This system has the solutions \( a = 2 + \sqrt{2} \) and \( b = 2 - \sqrt{2} \), and it is easy to check now directly that \( a \) and \( b \) have these values.

**Problem 6.21.** Find \( \sqrt[3]{5 + 2\sqrt{13}} + \sqrt[3]{5 - 2\sqrt{13}} \).

**Solution:** On the same lines as in the previous problem we find the value 1. Even more, the first cubic root is \( 1 + \frac{\sqrt{13}}{2} \) and the second one is \( 1 - \frac{\sqrt{13}}{2} \).

**Problem 6.22.** Prove that for any rational numbers \( a, b, c \) with \( c \geq 0 \) we have \( a + b\sqrt{c} \neq \sqrt{2} \).

**Problem 6.23.** If \( m, n \) are natural numbers such that \( \frac{m}{n} < \sqrt{7} \), then

\[
\frac{m}{n} + \frac{1}{mn} < \sqrt{7}.
\]

**Solution:** Assume there are \( m, n \in \mathbb{N} \) such that

\[
\frac{m}{n} < \sqrt{7} \leq \frac{m}{n} + \frac{1}{mn}
\]

Then after multiplication by \( n \) and squaring we obtain

\[
m^2 < 7n^2 \leq m^2 + 2 + \frac{1}{m^2} \leq m^2 + 3
\]
It follows that necessarily \( 7n^2 \) should be either \( m^2 + 1 \) or \( m^2 + 2 \). But for any integer \( m \) these numbers are not divisible by 7. Contradiction.

**Problem 6.24.** If \( 1 \leq a < b \), then there are positive integers \( m, n \) such that
\[
a < \sqrt[n]{b}.
\]

**Solution:** Since \( \lim_{n \to \infty} (b^n - a^n) = \infty \), there is \( n \) such that \( b^n - a^n > 1 \). Then there is an integer \( m \) in the interval \((a^n, b^n)\).

**Problem 6.25.** If \( q = 1 + \sqrt{5} \), then for every natural \( n \) we have \([q^n] + 1 = [q^{2n}]\).

**Solution:** Since \( q^2 = q + 1 \) and \([x + 1] = [x] + 1\), the equality to prove is equivalent with \([q^n] + 1 = [qn] + n\). Let \( m \) denote \([qn]\), which is by definition the only integer satisfying \( qn - 1 < m \leq qn \). Since \( q \notin \mathbb{Q} \), \( m \) is in fact the only integer with the property \( qn - 1 < m < qn \). We have to prove that \([qm] + 1 = m + n\), or equivalently \(qm < m + n \leq qm + 1\). Indeed, we obtain the left inequality in the following way:
\[
qm + 1 = m + (q - 1)m = m + \frac{m}{q} + 1 > m + n - \frac{1}{q} + 1 > m + n.
\]

**Problem 6.26.** Let \( x \) be a real number and \( m > 0 \) be an integer. Simplify
\[
[x] + \left[ x + \frac{1}{m} \right] + \ldots + \left[ x + \frac{m - 1}{m} \right].
\]

**Solution:** Let \( f : \mathbb{R} \to \mathbb{R} \) be the function defined by \( f(x) = [x] + \left[ x + \frac{1}{m} \right] + \ldots + \left[ x + \frac{m - 1}{m} \right] - [mx] \). The integer part has the property \([a + 1] = [a]\) for any real number \( a \). Using this we see that \( f(x + \frac{k}{m}) = f(x) \) for any \( x \). We remark also that \( \left[ x + \frac{k}{m} \right] = 0 \) for any \( x \in [0, \frac{1}{m}) \) and any \( k = 0, 1, 2, \ldots, m - 1 \). Thus \( f(x) = 0 \) for all \( x \in [0, \frac{1}{m}) \). But \( f \) has the period \( \frac{1}{m} \), so \( f(x) = 0 \) for any real \( x \), i.e.
\[
[x] + \left[ x + \frac{1}{m} \right] + \ldots + \left[ x + \frac{m - 1}{m} \right] = [mx].
\]

**Problem 6.27.** Determine all solutions of the system
\[
\begin{align*}
\frac{x + y + z}{z} &= w \\
\frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{w}
\end{align*}
\]

**Solution:** The second equation writes \( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x + y + z} \) and this equation is equivalent to
\[
\frac{(x + y)(y + z)(z + x)}{xyz(x + y + z)} = 0
\]
The solutions of the system are then of the form \((a, -a, b), (a, b, -a)\) and \((a, b, -b, a)\), where \(a, b\) are parameters.

**Problem 6.28.** Determine the triples of integers \((x, y, z)\) satisfying the equation \(x^3 + y^3 + z^3 = (x + y + z)^3\).

**Solution:** The identity \((x + y + z)^3 = x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x)\) shows the solutions of the equation are of the form \((a, -a, b), (a, b, -a)\) and \((a, b, -b)\).

**Problem 6.29.** Prove there is no arithmetic sequence which has between its terms the numbers \(\sqrt{2}, \sqrt{3}\) and \(\sqrt{5}\).

**Solution:** Supposing there is such a sequence of ratio \(r\), there are positive integers \(m, n\) such that \(\sqrt{3} - \sqrt{2} = mr\) and \(\sqrt{5} - \sqrt{3} = nr\). But then \(\frac{\sqrt{5} - \sqrt{3}}{\sqrt{3} - \sqrt{2}} = \frac{m}{n}\) is rational. Contradiction.

**Problem 6.30.**

a) If \(m, n\) are positive integers, prove that \(\frac{m + 2n}{m + n}\) is a better approximation of \(\sqrt{2}\) than \(\frac{m}{n}\).

b) Find the first 4 decimals of \(\sqrt{2}\).

**Solution:**

a) We have to show that \(\frac{m + 2n}{m + n} - \sqrt{2} < \frac{m}{n} - \sqrt{2}\), but this inequality can be easily written as

\[
\frac{|(m - n\sqrt{2})(1 - \sqrt{2})|}{m + n} < \frac{|m - n\sqrt{2}|}{n},
\]

or under the form

\[m + (2 - \sqrt{2})n > 0\]

b) We start with the approximation given by \(\frac{m_1}{n_1} = \frac{3}{2} = 1.5\). Then using a) we have a better approximation given by \(\frac{m_2}{n_2} = \frac{m_1 + 2n_1}{m_1 + n_1} = \frac{7}{5} = 1.4\). We continue with

\[
\begin{align*}
\frac{m_3}{n_3} & = \frac{m_2 + 2n_2}{m_2 + n_2} = \frac{17}{12} = 1.416666\ldots \\
\frac{m_4}{n_4} & = \frac{m_3 + 2n_3}{m_3 + n_3} = \frac{41}{29} = 1.413793\ldots \\
\frac{m_5}{n_5} & = \frac{m_4 + 2n_4}{m_4 + n_4} = \frac{99}{70} = 1.414285\ldots \\
\frac{m_6}{n_6} & = \frac{m_5 + 2n_5}{m_5 + n_5} = \frac{239}{169} = 1.414201\ldots \\
\frac{m_7}{n_7} & = \frac{m_6 + 2n_6}{m_6 + n_6} = \frac{577}{408} = 1.414215
\end{align*}
\]

**Problem 6.31.** Let \(k, m, n\) be positive integers such that \(\sqrt{k}\) is irrational and \(\frac{m}{n} > \sqrt{k} - 2\). Prove that \(\frac{m + nk}{m + n}\) is a better approximation of \(\sqrt{k}\) than \(\frac{m}{n}\).
CHAPTER 7

Complex numbers

PROBLEM 7.1. Let $n \geq 2$. Prove that $\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}$.

Solution: For each $k = \frac{1}{n}$ denote $x_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$ the $n$th-roots of unity different from 1 and we have

$$x^{n-1} + x^{n-2} + \ldots + x + 1 = \prod_{k=1}^{n-1} (x - x_k).$$

For $x = 1$ we obtain

$$n = \prod_{k=1}^{n-1} \left( 2 \sin^2 \frac{k\pi}{n} - 2i \sin \frac{k\pi}{n} \cos \frac{k\pi}{n} \right)$$

$$= 2^{n-1} \left( \cos \frac{(n-1)n\pi}{2n} + i \sin \frac{(n-1)n\pi}{2n} \right) (-i)^{n-1} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$$

$$= 2^{n-1} \prod_{k=1}^{n-1} \sin \frac{k\pi}{n}$$

PROBLEM 7.2. If $x_1, x_2, \ldots, x_n$ denote the $n$th roots of the unity, evaluate $N = \prod_{p<q} (x_p - x_q)^2$.

Solution: We have $x_p - x_q = 2i \sin \frac{(q-p)\pi}{n} \left( \cos \frac{(q+p)\pi}{n} + i \sin \frac{(q+p)\pi}{n} \right)$. Then

$$N = (-4)^{n(n-1)/2} \prod_{p<q} \sin^2 \frac{(q-p)\pi}{n} \left( \cos \frac{\sum_{q\neq p} (q+p)\pi}{n} + i \sin \frac{\sum_{q\neq p} (q+p)\pi}{n} \right).$$

Also

$$\sum_{q\neq p} (q+p) = \sum_{p=1}^{n} (q+p) = \sum_{p=1}^{n} \left( \frac{n(n+1)}{2} - p + p(n-1) \right) = n^2 - 1,$$

and

$$\prod_{p<q} \sin^2 \frac{(q-p)\pi}{n} = \left( \prod_{k=1}^{n} \sin \frac{k\pi}{n} \right)^n = \frac{n}{2^{n(n-1)}}.$$ Therefore $N = (-1)^{\frac{n(n-1)+2}{2}} n^n$.

PROBLEM 7.3. (i) Let $z_1, z_2, \ldots, z_n$ be complex numbers. Then there is a subset $J$ of $\{1, 2, \ldots, n\}$ such that $|\sum_{j \in J} z_j| \geq \frac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j|$. 
Since the two bisectors of the complex plane determine the following four sets

\(A_1 = \{ z \in \mathbb{C} | \text{Re} \, z \geq |\text{Im} \, z| \}, \ A_2 = \{ z \in \mathbb{C} | -\text{Re} \, z \geq |\text{Im} \, z| \}\)

\(A_3 = \{ z \in \mathbb{C} | \text{Im} \, z \geq |\text{Re} \, z| \}, \ A_4 = \{ z \in \mathbb{C} | -\text{Im} \, z \geq |\text{Re} \, z| \}\)

For at least one of these sets, say \(A_1\) (for the other sets the proof is similar)

\[\sum_{j \in J} |z_j| \geq \frac{1}{4} \sum_{j=1}^{n} |z_j|, \text{ Let } J = \{j|z_j \in A_1\}. \text{ Remark that for } j \in J, \ |z_j| \leq \sqrt{2} \text{Re} \, z_j,\]

and then

\[\sum_{j \in J} |z_j| \geq \sum_{j \in J} \text{Re} \, z_j \geq \sum_{j \in J} \frac{1}{\sqrt{2}} |z_j| \geq \frac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j| .\]

(ii) We can suppose without costs that \(z_1, z_2, ..., z_k \geq 0\) and \(z_{k+1}, ..., z_n \leq 0\).

Since

\[\sum_{j=1}^{k} z_j + \sum_{j=k+1}^{n} z_j \geq \sum_{j=1}^{n} |z_j|,\]

we can take \(J = \{1, 2, ..., k\}\) if \(\sum_{j=1}^{k} z_j \geq \sum_{j=k+1}^{n} z_j\) and \(J = \{k+1, ..., n\}\) otherwise.

**Problem 7.4.** Let \(D = \{z \in \mathbb{C}, |z| \leq 1\}\) and \(f : D \to D\) a function such that \(|f(z_1) - f(z_2)| \geq |z_1 - z_2|\) for all \(z_1, z_2 \in D\). Then there is \(z_0 \in D\) such that \(f(z_0) = z_0\).

**Solution:** Consider the points \(u, -u, v, -v\) on the complex unit circle such that \(\text{Re} \, u = \text{Im} \, v\) and \(\text{Im} \, u = \text{Re} \, v\). This means the points \(u, v, -u, -v\) are the vertices of a square. From the inequalities \(2 = |u - (-u)| \leq |f(u) - f(-u)| \leq |f(u)| + |f(-u)| \leq 2\) and \(2 = |v - (-v)| \leq |f(v) - f(-v)| \leq |f(v)| + |f(-v)| \leq 2\) we deduce that \(f(u), f(v), f(-u), f(-v)\) are on the complex unit circle and also \(|f(u) - f(-u)| = |f(v) - f(-v)| = 2\). Therefore the segments \((f(u), f(-u))\) and \((f(v), f(-v))\) are diameters. From \(\sqrt{2} = |u - v| \leq |f(u) - f(v)|\) and \(\sqrt{2} = |u - (-v)| \leq |f(u) - f(-v)|\) we see these diameters are orthogonal, so \(f(u), f(v), f(-u), f(-v)\) are the vertices of a square. Remark now that the origin \(0\) is in \(D\), hence \(|f(0)| \leq 1\) and \(|f(0) - f(u)| \geq |0 - u| = 1\), \(|f(0) - f(v)| \geq |0 - v| = 1\), \(|f(0) - f(-u)| = |0 + u| = 1\), \(|f(0) - f(-v)| \geq |0 + v| = 1\). Then \(f(0)\) must be 0 as the only point in the interior of the unit circle with the distances to the vertices of an inscribed square greater or equal than 1.

**Problem 7.5.** Let \(a, b, c\) be complex numbers on the unit circle in the complex plane. Then \(a, b, c\) are vertices of an equilateral triangle if and only if \(a + b + c = 0\).

**Solution:** Suppose \(a + b + c = 0\). The equality \(|a + b| = |a - c| = 1\) can be written \((a + b)(\bar{a} + \bar{b}) = 1\). But \(\frac{a + b}{2} = 1\) so \(ab + \bar{a}b = -1\). Therefore \(|a - b|^2 = (a - b)(\bar{a} - \bar{b}) = 3\). In this way we prove that \(|a - b| = |a - c| = |b - c| = \sqrt{3}\) and \(a, b, c\) are the vertices of an equilateral triangle.

Reciprocally, suppose \(a, b, c\) are the vertices of an equilateral triangle. Then there is a \(t\) such that \(a = \cos t + i \sin t\), \(b = \cos(t + \frac{2\pi}{3}) + i \sin(t + \frac{2\pi}{3})\), \(c = \cos(t + \frac{4\pi}{3}) + i \sin(t + \frac{4\pi}{3})\).
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\[ i \sin(t + \frac{4\pi}{3}) \]. Using \( \cos t + \cos(t + \frac{2\pi}{3}) + \cos(t + \frac{4\pi}{3}) = \cos(t + \frac{2\pi}{3})(2 \cos \frac{2\pi}{3} + 1) = 0 \)
and \( \sin t + \sin(t + \frac{2\pi}{3}) + \sin(t + \frac{4\pi}{3}) = \sin(t + \frac{2\pi}{3})(2 \cos \frac{2\pi}{3} + 1) = 0 \) we get \( a + b + c = 0 \).

**Problem 7.6.** If \( a \) and \( b \) are complex numbers with the real part negative or zero then \( |e^a - e^b| \leq |a - b| \).

**Solution:** If \( a = x + iy \) and \( b = u + iv \) then

\[ |e^a - e^b|^2 = (e^{2x} - e^{2u})^2 + 4e^{x+u} \sin^2 \frac{y-v}{2}. \]

There is \( t \) between \( x \) and \( u \), hence negative, such that \( e^x - e^u = (x-u)e^t \). Therefore,

\[ |e^a - e^b|^2 \leq |x-u|^2 + |y-v|^2 = |a - b|^2. \]

**Problem 7.7.** Prove that if \( \lambda > 1 \), the function \( f(z) = z + \lambda - e^z \) has only one zero in the half-plane \( \text{Re} z < 0 \), and this zero is on the real axis.

**Solution:** Let \( z = x + iy \), \( x, y \) reals, \( x < 0 \) be a zero of the function \( f \). Then \( e^x \cos y - x = \lambda \) and \( e^x \sin y = y \). From \( |y| = e^x \sin y \leq |\sin y| \) we see that \( y = 0 \) and consequently \( x = \lambda \). Since the function \( \varphi(x) = e^x - x \) is strictly decreasing on the interval \((-\infty, 0)\) and \( \varphi(0) = 1 \), the equation \( \varphi(x) = \lambda \) has exactly one solution.

**Problem 7.8.** If 4 distinct complex numbers have the property that the product of any 3 of them is the fourth number, then they are the vertices of a square in the complex plane.

**Solution:** We can suppose without any cost that all four numbers are non-zero, otherwise if one is zero then necessarily all are equal zero. The four numbers, say \( x, y, z, t \), satisfy the system \( xyz = t^3 \), \( yzt = x^3 \), \( ztx = y^3 \), \( txy = z^3 \). Dividing each two successive equations we obtain \( \left( \frac{x}{t} \right)^4 = \left( \frac{y}{t} \right)^4 = \left( \frac{z}{t} \right)^4 = 1 \). Since the numbers are distinct we have \( x = q_1t \), \( y = q_2t \), \( z = q_3t \), where \( q_1, q_2, q_3 \) are the roots different of 1 of the equation \( t^4 - 1 = 0 \). But then \( x, y, z, t \) lie on a circle centered in origin with radius \(|t| \) and are the vertices of a square.

**Problem 7.9.** In any regular polygon with an odd number of vertices any two sides are not parallel.

**Solution:** Assume there is such a polygon. Without loss of generality we can assume the polygon inscribed in the unit circle in the complex plane and with the vertices given by the

\[ z_k = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \quad \text{for } k = 0, 1, \ldots n - 1. \]

The slope of the segment passing through the vertices \( z_k \) and \( z_{k+1} \) is

\[ \frac{\sin \frac{2(k+1)\pi}{n} - \sin \frac{2k\pi}{n}}{\cos \frac{2k\pi}{n} - \cos \frac{2(k+1)\pi}{n}} = -\cot \frac{(2k + 1)\pi}{n}. \]
By our assumption there exist $0 \leq k < l \leq n - 1$ such that $-\cot \frac{(2k + 1)\pi}{n} = -\cot \frac{(2l + 1)\pi}{n}$. But this means $\frac{(2k + 1)\pi}{n} - \frac{(2l + 1)\pi}{n} = \pi$, or $2(k - l) = n$. Contradiction with the hypothesis that $n$ is odd.

**Problem 7.10.** Prove that for any non-zero complex number $z$, the number $\frac{z}{|z|} + \frac{|z|}{z}$ is real.

**Solution:** If $z = |z|(\cos t + i \sin t)$ then $\frac{z}{|z|} + \frac{|z|}{z} = 2 \cos t$.

**Problem 7.11.** Prove that for any complex number $z$ there exists real numbers $a, b$ such that $z^2 = az + b$.

**Solution:** Using the previous problem we have $\frac{z}{|z|} + \frac{|z|}{z}$. Take $a = 2|z| \cos t$ and $b = -|z|^2$.

**Problem 7.12.** Prove that for any complex numbers $u, v$ we have the parallelogram law $|u - v|^2 + |u + v|^2 = 2(|u|^2 + |v|^2)$.

**Solution:** Use $z \cdot \bar{z} = |z|^2$.

**Problem 7.13.** Let $m, n$ be relatively prime positive integers. Then the complex equations $z^n - 1 = 0$ and $z^m - 1 = 0$ have exactly one common root.

**Solution:** A solution of the first equation has the form $z_k = e^{\frac{2k\pi}{n}}$ with $k \in \{0, 1, 2, \ldots, n - 1\}$ and a solution of the second equation has the form $z_l = e^{\frac{2l\pi}{m}}$ with $l \in \{0, 1, 2, \ldots, m - 1\}$. Then $z_k = z_l$ corresponds to $\frac{2k\pi}{n} = \frac{2l\pi}{m}$ or $km = ln$. Since $m, n$ are relatively prime it follows that $n$ divides $k < n$ which is possible only for $k = 0$. Similarly we obtain $l = 0$ which give the only common zero $1$.

**Problem 7.14.** Consider the real numbers $a > b > 0$ and the complex number $z$ such that $|z - a| = \sqrt{a^2 - b^2}$.

**Solution:** Prove that $\frac{b - z}{b + z} = \sqrt{a - b}$. Then $\frac{b - z}{b + z} = \frac{b - z}{b + z} = \frac{b^2 - b\bar{z} - b^2 + az + a\bar{z} - b^2}{b^2 + b\bar{z} + b^2 + az + a\bar{z} - b^2} = \frac{a - b}{a + b}$.

**Problem 7.15.** Let $u, v, w$ be complex numbers such that $|u| = |v| = |w| = 1$, $u^2 + v^2 + w^2 = 0$, and $u + v + w \neq 0$. Prove that $|u + v + w| = 2$.

**Solution:** We have $1 = |w|^2 = |u - u^2 - v^2|^2 = (u^2 + v^2)(\bar{u}^2 + \bar{v}^2) = (u^2 + v^2)\left(\frac{1}{u^2} + \frac{1}{v^2}\right) = 2 + \frac{u^2}{v^2} + \frac{v^2}{u^2}$. Hence $0 = u^4 + u^2v^2 + v^4 = (u^2 + uv + v^2)(u^2 - uv + v^2) = (uv - w^2)(-uv - w^2) = w^4 - u^2v^2$. Similarly we obtain $0 = v^4 - u^2w^2$. 

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Then $u^2 = \frac{w^4}{v^2} = \frac{v^4}{w^2}$ and from here $v^6 = w^6$. There exist a complex number $a$ such that $a^6 = 1$ and $v = aw$. Then $u^2 = \frac{v^4}{w^2} = a^4w^2$, so $u = \pm a^2w$. It follows that

$$|u + v + w| = |w||1 + a \pm a^2| = 2.$$  Indeed $a = e^{\frac{2k\pi}{6}}$ with $k \in \{1, 3, 5\}$, otherwise we would have $u + v + w = 0$. 

CHAPTER 8

Equations

Problem 8.1. a) Solve the system
\[
\begin{align*}
\frac{bc}{x} + \frac{ca}{y} + \frac{ab}{z} &= 1 \\
\frac{b+c}{x} + \frac{c+a}{y} + \frac{a+b}{z} &= 0 \\
\frac{xy}{x} + \frac{yx}{y} + \frac{zx}{z} &= 0
\end{align*}
\]

b) If \(a, b, c\) are the sides of a triangle then \(x + y + z < ab + bc + ca\).

Solution : a) Taking \(u = \frac{1}{x}, v = \frac{1}{y}, t = \frac{1}{z}\) the system becomes
\[
\begin{align*}
bcu + cav + abt &= 1 \\
au + bv + ct &= 0 \\
u + v + t &= 0
\end{align*}
\]
and has the solution \(u = \frac{1}{(a-b)(a-c)}, v = \frac{1}{(b-a)(b-c)}, t = \frac{1}{(c-a)(c-b)}\).

Hence \(x = (a-b)(a-c), y = (b-c)(b-a), z = (c-a)(c-b)\).

b) The inequality is equivalent with \(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca < 0\), or \((\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{b} + \sqrt{c} - \sqrt{a})(\sqrt{b} - \sqrt{c} - \sqrt{a})(\sqrt{b} - \sqrt{c} + \sqrt{a}) < 0\). And this is a consequence of the fact that in a triangle we have \(\sqrt{b} + \sqrt{c} > \sqrt{a}\) and the other similar inequalities.

Problem 8.2. Let \(a\) be a fixed real number. Find the solutions of the system
\[
\begin{align*}
x_1^2 + ax_1 + \frac{(a-1)^2}{4} &= x_2 \\
x_2^2 + ax_2 + \frac{(a-1)^2}{4} &= x_3 \\
\ldots &\ldots \\
x_n^2 + ax_n + \frac{(a-1)^2}{4} &= x_1
\end{align*}
\]

Solution : The sum of all equations can be arranged \(\sum_{k=1}^{n} \left(x_k - \frac{a-1}{2}\right)^2 = 0\).

Hence \(x_1 = x_2 = \ldots = x_n = \frac{a-1}{2}\).

Problem 8.3. Solve the equation \(2x^2 + |x| = x^4\).

Solution : From \(x + 1 \geq |x| + 1 = (x^2 - 1)^2\) we see \(x \geq -1\). We examine the cases.

A) \(x \in [-1, 0)\). Then \(|x| = -1\) and the equation is \((x^2 - 1)^2 = 0\), with the only solution \(x = -1\).
B) $x \geq 0$. The inequality $x^4 - 2x^2 = [x] \leq x$ can be arranged $(x+1)(x^2-x-1) \leq 0$ so necessarily $x \in \left[ 0, \frac{1+\sqrt{5}}{2} \right]$. For $x \in [0, 1)$ the equation is $x^4 = 2x^2$ with the only solution $x = 0$, and for $x \in \left[ 1, \frac{1+\sqrt{5}}{2} \right]$, the equation becomes $(x^2 - 1)^2 = 2$ with the only solution $x = \sqrt{1+\sqrt{2}}$.

**Problem 8.4.** Solve the equation

$$\sum_{k=1}^{n} \frac{1}{\cos x + \cos(2k+1)x} = \sum_{k=1}^{n} \frac{1}{\cos x - \cos(2k+1)x}$$

**Solution:** We will examine two cases. Suppose first $\sin x \neq 0$. Then

$$\sum_{k=1}^{n} \frac{1}{\cos x + \cos(2k+1)x} = \frac{1}{2\sin x} \sum_{k=1}^{n} \frac{\sin((k+1)x - kx)}{\cos(k+1)x \cos kx}$$

$$= \frac{1}{2\sin x} \sum_{k=1}^{n} (\tan(k+1)x - \tan kx)$$

$$= \frac{\sin nx}{2\sin x \cos x \cos(n+1)x}$$

Similar, $\sum_{k=1}^{n} \frac{1}{\cos x - \cos(2k+1)x} = \frac{\sin nx}{2\sin x \sin x \sin(n+1)x}$. The equation then is $\sin nx \cos(n+2)x = 0$, with solutions $x = \frac{p\pi}{n}$, where $p/n$ is not an integer and $x = \frac{(2p+1)\pi}{2(n+2)}$.

If $\sin x = 0$, then the right hand side of the equation is not defined.

**Problem 8.5.** Solve the equation $\sqrt{x} + \sqrt{y-1} + \sqrt{z-2} = \frac{x+y+z}{2}$.

**Solution:** With the notations $\sqrt{x} = a$, $\sqrt{y-1} = b$ and $\sqrt{z-2} = c$, the equation becomes $(a-1)^2 + (b-1)^2 + (c-1)^2 = 0$. The only solution in real numbers is $x = 1$, $y = 2$, $z = 3$.

**Problem 8.6.** Solve in positive numbers the system $x^5y = 64$, $5x + y = 12$.

**Solution:** By the inequality between the arithmetic mean and the geometric mean $12 = 5x + y \geq 6\sqrt{x^5y} = 12$. The equality holds only when $x = y = 2$.

One can solve the system without the assumption about the positivity. From the first equation $y = 12 - 5x$ and substituting in the second one we get the equation $5x^6 - 12x^5 + 64 = (x - 2)^2(5x^4 + 8x^3 + 12x^2 + 16x + 16) = 0$. But $5x^4 + 8x^3 + 12x^2 + 16x + 16 = 3x^4 + 2(x^4 + 4x^3 + 4x^2) + 4(x^2 + 4x + 4) > 0$.

**Problem 8.7.** Let $a \in (0, 1)$ be fixed. Prove that the equation $x^a = a^x$ has a unique solution.
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Solution : Remark first that \( x \in (0, 1) \). The equation is equivalent with \( f(x) = \ln \ln x + x \ln a - a \ln x - \ln \ln a = 0 \). Since \( f(a) = 0 \) it suffices to prove that \( f \) is injective on \((0, 1)\). But \( f'(x) = \frac{1}{x \ln x} + \ln a - \frac{a}{x} < 0 \), so \( f \) is strictly decreasing.

Problem 8.8. Solve the equation \( 2^{\log_3 x} + 3^{\log_x 2} = 4 \).

Solution : Necessarily \( x > 0 \) and \( x \neq 1 \). For \( x \in (0, 1) \) both exponents are negative so \( 2^{\log_3 x} + 3^{\log_x 2} < 2 \). If \( x > 1 \), then \( 2^{\log_3 x} + 3^{\log_x 2} = 2^{\log_3 x} + 2^{\log_x 3} \geq 2 \sqrt{2^{\log_3 x} \cdot 2^{\log_x 3}} = 2 \sqrt{2} = 4 \). The equality takes place when \( \log_3 x = \log_x 3 = \frac{1}{\log_3 x} \), so \( x = 3 \) is the only solution.

Problem 8.9. Find all pairs of real numbers \((x, y)\) satisfying the system of equations

\[
\begin{align*}
\frac{1}{x} + \frac{1}{2y} &= (x^2 + 3y^2)(3x^2 + y^2) \\
\frac{1}{x} - \frac{1}{2y} &= 2(y^4 - x^4).
\end{align*}
\]

Solution : We add the equations, eliminate the denominator, divide by 2, respectively we subtract the equations and eliminate the denominator, to get the first, respectively the second equation of the equivalent system

\[
\begin{align*}
x^5 + 10x^3y^2 + 5xy^4 &= 2 \\
5x^4y + 10x^2y^3 + y^5 &= 1
\end{align*}
\]

Adding and subtracting these two equations we obtain

\[
\begin{align*}
(x + y)^5 &= 3 \\
(x - y)^5 &= 1
\end{align*}
\]

Therefore \( x = \frac{\sqrt{3} + 1}{2} \) and \( y = \frac{\sqrt{3} - 1}{2} \).

Problem 8.10. Curves \( A, B, C \) and \( D \) are defined in the plane as follows:

\[
\begin{align*}
A &= \left\{ (x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2} \right\}, \\
B &= \left\{ (x, y) : 2xy + \frac{y}{x^2 + y^2} = 3 \right\}, \\
C &= \left\{ (x, y) : x^3 - 3xy^2 + 3y = 1 \right\}, \\
D &= \left\{ (x, y) : 3x^2y - 3x - y^3 = 0 \right\}.
\end{align*}
\]

Prove that \( A \cap B = C \cap D \). [P1987]

Solution : Let \((x, y) \in A \cap B\). Then \((x, y) \neq (0, 0)\) and the system can be written

\[
\begin{align*}
(x^2 + y^2)(x^2 - y^2) &= x \\
(x^2 + y^2)(3 - 2xy) &= y
\end{align*}
\]

Taking the quotient of this two equations and rearranging we obtain \((x, y) \in D\). We remark now that \( x \neq 0 \), since otherwise from \( y^3 = 3x^2y - 3x \) we obtain \( y = 0 \), and \((x, y) = (0, 0)\). Multiplying with \( y \) the equation representing the fact \((x, y) \in D\) we get \( y^4 = 3x^2y^2 - 3xy \) which combined with \((x, y) \in A\) and after we cancel \( x \) gives \((x, y) \in C\).
Let now \((x, y) \in C \cap D\). First we remark that \(x^2 + y^2 \neq 0\). Indeed, \((0, 0) \notin C\). We multiply with \(x\) the equation which defines \((x, y) \in C\), with \(y\) the one representing \((x, y) \in D\), we add them and we get

\[
(x^2 - y^2)(x^2 + y^2)
\]

We can divide by \(x^2 + y^2\) and we get \((x, y) \in A\). We can write \((x, y) \in D\) as

\[
(x^2 - y^2)y = x(3 - 2xy)
\]

We notice that \(x^2 - y^2 \neq 0\), since if we suppose the contrary, then from 0.1 we obtain \(x = 0\) and \((0, y) \in D\) implies \(y = 0\), so \(x^2 + y^2 = 0\).

We multiply the equations 0.1 and 0.2, we cancel \((x^2 - y^2)\) and rearranging we obtain \((x, y) \in B\).

**Problem 8.11.** Let \(n \geq 2\) be an integer. Solve the equation \(\sin^n x - \cos^n x = 1\).

**Solution:** The equation can be written as \(\sin^n x - \cos^n x = \sin^2 x + \cos^2 x\), thus in \(\sin^n x \leq \sin^2 x\) and \(-\cos^n x \leq \cos^2 x\) we must have equalities. If \(n\) is even then \(\cos x = 0\) and \(x = 2\pi k \pm \frac{\pi}{2}\), with \(k\) integer is a solution. If \(n\) is odd, \(\cos x = 0\) and \(\sin x = 1\), and \(x = 2\pi k + \frac{\pi}{2}\) is a solution, and also we have the case \(\cos x = -1, \sin x = 0\), with \(x = (2k + 1)\pi\) solution.

**Problem 8.12.** Solve the equation

\[
\sin^2 x + \sin 2x \sin 4x + \ldots + \sin nx \sin n^2 x = 1.
\]

**Solution:** We have

\[
1 = \sum_{k=1}^{n} \sin kx \sin k^2 x = \frac{1}{2} \sum_{k=1}^{n} (\cos(k - 1)kx - \cos(k + 1)x)
\]

\[
= \frac{1}{2} \left( 1 - \cos(n^2 + n)x \right).
\]

The solution is \(x = \frac{(2k + 1)\pi}{n^2 + n}\).

**Problem 8.13.** Let \(n\) be a fixed positive integer. How many rational solutions has the equation \(\log_{2^n} x = 2^{n^2}\)?

**Solution:** The equation can be rewritten \(2^{n^2} x = x\). We prove this equation doesn’t have any real root. First, we note \(x > 0\). Then \(x = 2^{n^2} x > 2^n\). Again using this inequality we obtain \(x = 2^n 2^{n^2} > 2^{n^2} 2^n\), and by recurrence we see that the root \(x\) has to be higher than any real number. Contradiction.

**Problem 8.14.** Solve the equation \(2^{\sin x} + 2^{\cos x} = \sqrt{2^{2\sin x} - \sqrt{2}}\).

**Solution:** We have \(2^{\sin x} + 2^{\cos x} \geq 2^{\sqrt{2}\sin x}, 2^{\frac{\cos x}{2}} = \sqrt{2^{2\sin x - \cos x}} \geq \sqrt{2^{2\frac{\cos x}{2}}}\) and the equality takes place for \(\sin x = \cos x = -\frac{\sqrt{2}}{2}\).

**Problem 8.15.** a) Solve in real numbers the equation \(2^x + 4^x = 3^x + 5^x\).

b) Solve in real numbers the equation \(2^x + 5^x = 3^x + 4^x\).
Solution: a) For $x > 0$, $3^x + 5^x > 2^x + 4^x$, and for $x < 0$, $3^x + 5^x < 2^x + 4^x$, so $x = 0$ is the only solution.

b) The mean value theorem for the function $f(t) = t^x$ shows there is $c \in (2, 3)$ and $d \in (4, 5)$ such that $3^x - 2^x = x c^{x-1}$ and $5^x - 4^x = x d^{x-1}$. The equation becomes $x \left[ \left( \frac{d}{c} \right)^{x-1} - 1 \right] = 0$, and has the solutions $x = 0$ and $x = 1$.

Problem 8.16. Let $n$ be a fixed positive integer. How many rational solutions has the equation $\log_{2^n} x = 2^{nx}$.

Solution: Consider the functions $f : (0, \infty) \to \mathbb{R}$, $f(x) = \log_{2^n} x$ and $g : \mathbb{R} \to (0, \infty)$, $g(x) = 2^{nx}$. We observe that $g = f^{-1}$, so the equation can be written $f(x) = f^{-1}(x)$. But a function can be equal with its inverse only on the line $y = x$, so the equation is equivalent with $2^{nx} = x$.

Since for any $x > 0$, we have $2^{nx} \geq 2^x > x$, the equation doesn’t have any real solution.
1. Jensen’s inequality

**Definition 9.1.** A function $f : (a, b) \to \mathbb{R}$ is called convex (concave) if for any $x, y \in (a, b)$ and any $t \in [0, 1]$ the following inequality holds

$$f(tx + (1 - t)y) \leq (\geq) tf(x) + (1 - t)f(y)$$

For $\alpha < \beta$ we have $[\alpha, \beta] = \{t\alpha + (1 - t)\beta, t \in [0, 1]\}$. Then the function $f$ is convex (concave) if for any two points $x < y$, the graph of the function is under (above) the line determined by the points $(x, f(x))$ and $(y, f(y))$.

A function is convex if and only if $-f$ is concave.

**Proposition 9.2.** If a function $f$ is two times derivable on the interval $(a, b)$, then $f$ is convex (concave) if and only if $f'' \geq (\leq) 0$ on $(a, b)$.

**Jensen’s inequality** If $f : (a, b) \to \mathbb{R}$ is a convex (concave) function then for any $x_1, x_2, ..., x_n \in (a, b)$ and any $t_1, t_2, ..., t_n$ positive numbers with $\sum_{k=1}^{n} t_k = 1$ the following inequality holds

$$f(t_1x_1 + t_2x_2 + ... + t_nx_n) \leq (\geq) t_1f(x_1) + t_2f(x_2) + ... + t_nf(x_n)$$

**Problem 9.1.** If $A, B, C$ are the angles of a triangle then

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$ 

**Solution:** The function $f(x) = \sin x$ is concave on the interval $[0, \pi]$, hence by Jensen’s inequality we have

$$\frac{\sin A + \sin B + \sin C}{3} \leq \sin \frac{A + B + C}{3} = \frac{\sqrt{3}}{2}.$$ 

**Problem 9.2.** If $A, B, C$ are the angles of an acute triangle then

$$\cos A + \cos B + \cos C \leq \frac{3}{2}.$$ 

**Solution:** The function $f(x) = \cos x$ is concave on the interval $[0, \frac{\pi}{2}]$, hence by Jensen’s inequality we have

$$\frac{\cos A + \cos B + \cos C}{3} \leq \cos \frac{A + B + C}{3} = \frac{1}{2}.$$
Second solution: The left hand side of the inequality can be arranged as
\[
\text{LHS} = 2 \cos \frac{A + B}{2} \cos \frac{A - B}{2} + 1 - 2 \sin^2 \frac{C}{2}
\]
\[
= -2 \sin^2 \left( \frac{C}{2} \right) + 2 \sin \left( \frac{A - B}{2} \right) - \frac{1}{2} \left( \frac{1}{2} + \frac{1}{2} \right)
\]
\[
= -\frac{1}{2} \left( 2 \sin \left( \frac{C}{2} \right) - \frac{1}{2} \left( \frac{A - B}{2} \right) \right)^2 - \frac{1}{2} \sin^2 \frac{A - B}{2} + 3 \geq \frac{3}{2}
\]

Problem 9.3. If \( A, B, C \) are the angles of an acute triangle then
\[
\tan A + \tan B + \tan C \geq 3\sqrt{3}.
\]
Solution: The function \( f(x) = \tan x \) is convex over the interval \( \left(0, \frac{\pi}{2}\right)\), thus by Jensen’s inequality we have
\[
\frac{\tan A + \tan B + \tan C}{3} \geq \tan \frac{A + B + C}{3} = \sqrt{3}
\]

Problem 9.4. If \( m, n \) are positive integers and \( x_1, x_2, \ldots, x_n \) are nonnegative real numbers, then
\[
\frac{x_1^m + x_2^m + \ldots + x_n^m}{n} \geq \left( \frac{x_1 + x_2 + \ldots + x_n}{n} \right)^m
\]
Solution: For any positive integer \( m \) the function \( f(x) = x^m \) is convex over the interval \( (0, \infty) \). We use Jensen’s inequality and we get
\[
\frac{x_1^m + x_2^m + \ldots + x_n^m}{n} \geq \left( \frac{x_1 + x_2 + \ldots + x_n}{n} \right)^m
\]

Problem 9.5. If \( a + b = 2 \) then \( a^4 + b^4 \geq 2 \).
Solution: We observe that the function \( f(x) = x^4 \) is convex over \( \mathbb{R} \), hence by Jensen’s inequality we have
\[
\frac{a^4 + b^4}{2} \geq \left( \frac{a + b}{2} \right)^4 = 1
\]
Second solution: We solve for \( b = 2 - a \) and we substitute into
\[
a^4 + b^4 = a^4 + (2 - a)^4 = 2a^4 - 8a^3 + 24a^2 - 32a + 16
\]
\[
= 2(a - 1)^2(a^2 - 2a + 7) + 2 \geq 2
\]

2. AM \( \geq \) GM \( \geq \) HM

Let \( x_1, x_2, \ldots, x_n \) be positive real numbers. We define their
- arithmetic mean \( AM(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \ldots + x_n}{n} \)
2. AM ≥ GM ≥ HM

- geometric mean $GM(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1x_2\cdots x_n}$

- harmonic mean $HM(x_1, x_2, \ldots, x_n) = \frac{1}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$

The Inequality of Means $AM \geq GM \geq HM$, and the equality holds if and only if $x_1 = x_2 = \ldots = x_n$.

Proof: Using Jensen’s inequality for the concave function $f(x) = \ln x$ and $t_1 = t_2 = \ldots = t_n = \frac{1}{n}$ we obtain $AM \geq GM$. Then taking $x_k \to \frac{1}{x_k}$ for $k = 1, 2, \ldots, n$ in $AM \geq GM$ we obtain $GM \geq HM$.

If $x_1, x_2, \ldots, x_n, a, b, c$ are positive numbers prove the following inequalities. Determine when the equalities hold.

**Problem 9.6.** If $x_1, x_2, \ldots, x_n$ are positive numbers prove that

$$(x_1 + x_2 + \ldots + x_n) \left( \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n} \right) \geq n^2.$$

Determine when the equality holds.

**Solution:** The inequality to prove corresponds to

$$AM(x_1, x_2, \ldots, x_n) \geq HM(x_1, x_2, \ldots, x_n),$$

and the equality holds when all the numbers are equal.

**Problem 9.7.** If $x_1, x_2, \ldots, x_n$ are positive numbers such that $x_1x_2\ldots x_n = 1$, then

$$(1 + x_1)(1 + x_2)\ldots(1 + x_n) \geq 2^n.$$

**Solution:** For each $i \in \{1, 2, \ldots, n\}$ we have $1 + x_i \geq 2\sqrt{x_i}$ with equality for $x_i = 1$. We multiply all these inequalities and we obtain the inequality to prove. The equality will hold when $x_1 = x_2 = \ldots = x_n = 1$.

**Problem 9.8.** If $x_1, x_2, \ldots, x_n$ are positive numbers then

$$\frac{x_1}{x_2} + \frac{x_2}{x_3} + \ldots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n.$$

1. $(a + b)(b + c)(c + a) \geq 8abc$

2. $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$

3. $\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} \leq \frac{a+b+c}{2}$

4. $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right)$

5. If $a, b, c$ are nonnegative real numbers, then $9abc \leq (a + b + c)(ab + bc + ca)$.

6. If $a, b, c$ are the sides of a triangle, then $(a-b+c)(b-c+a)(c-a+b) \leq abc$. 
7. In \(A, B, C\) are the angles (in radians) of a triangle, then
\[8 \cos A \cos B \cos C \leq 1.\]

8. Find the minimum value of \(2x + 3y\), knowing that \(x, y\) are positive numbers such that \(x^2y^3 = 1\).

9. If \(a_1, a_2, ..., a_n\) are positive real numbers with \(S = \sum_{k=1}^{n} a_k\), prove that
\[
\frac{a_1}{S - a_1} + \frac{a_2}{S - a_2} + ... + \frac{a_n}{S - a_n} \geq \frac{n}{n - 1}
\]

10. If \(a, b, c, d\) are positive real numbers with \(a + b + c + d = 1\), find the minimum value of \(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\).

11. If \(a, b, c, x, y, z\) are positive real numbers, then
\[
3\sqrt[3]{(a + x)(b + y)(c + z)} \geq 3\sqrt[3]{abc} + 3\sqrt[3]{xyz}
\]

3. The Cauchy-Schwarz Inequality

If \(a_1, a_2, ..., a_n, b_1, b_2, ..., b_n\) are real numbers, then
\[
(a_1^2 + a_2^2 + ... + a_n^2)(b_1^2 + b_2^2 + ... + b_n^2) \geq (a_1b_1 + a_2b_2 + ... + a_nb_n)^2
\]
and the inequality holds if and only if \(\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}\).

**Proof 1:** Induction after \(n\).

**Proof 2:** Immediate consequence of Lagrange’s identity
\[
\sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_i^2 - \left( \sum_{i=1}^{n} a_i b_i \right)^2 = \sum_{i<j} (a_i b_j - a_j b_i)^2
\]

**Proof 3:** Consider the function \(f(x) = (a_1^2 + a_2^2 + ... + a_n^2)x^2 - 2(a_1b_1 + a_2b_2 + ... + a_nb_n)x + (b_1^2 + b_2^2 + ... + b_n^2)\). Since \(f(x) = \sum_{k=1}^{n} (a_kx - b_k)^2 \geq 0\), the discriminant is negative, and this gives exactly the Cauchy-Schwarz inequality.

Applications:

1. \(AM \geq HM\)

2. (Minkowski) \(\sqrt{\sum_{i=1}^{n} (a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^{n} a_i^2} + \sqrt{\sum_{i=1}^{n} b_i^2}\)

3. \(\left( \frac{a_1 + a_2 + ... + a_n}{n} \right)^2 \leq \frac{a_1^2 + a_2^2 + ... + a_n^2}{n}\)

4. \(\sqrt{(a + b)(x + y)} \geq \sqrt{ax} + \sqrt{by}\)
5. Determine \( \min\{x^2 + y^2 + z^2 | x + 2y + 3z = 1\} \).

6. \[ \sqrt{a-c} + \sqrt{b-c} \leq \frac{a+b}{c} \]

7. If \( x^2 + a^2 + b^2 + c^2 + d^2 = 16 \) and \( x + a + b + c + d = 8 \), find the minimum and maximum value of \( x \).

8. \[ \sqrt{x+y} \geq x\sqrt{1-y} + y\sqrt{1-x} \], for any \( x, y \in [0,1] \)

9. Determine the minimum positive integer \( n \), such that for any real numbers \( x, y, z \) we have

\[ (x^2 + y^2 + z^2)^2 \leq n(x^4 + y^4 + z^4) \]

### 4. Chebyshev’s Inequality

If \( a_1, a_2, ..., a_n \) and \( b_1, b_2, ..., b_n \) are monotonic families of real numbers, then

\[ \frac{a_1 + a_2 + ... + a_n}{n} \frac{b_1 + b_2 + ... + b_n}{n} \leq \frac{a_1 b_1 + a_2 b_2 + ... + a_n b_n}{n} \]

if they have the same monotonicity (both increasing, or both decreasing), and

\[ \frac{a_1 + a_2 + ... + a_n}{n} \frac{b_1 + b_2 + ... + b_n}{n} \geq \frac{a_1 b_1 + a_2 b_2 + ... + a_n b_n}{n} \]

if the families have different monotonicity.

**Proof:** Without any restriction of the generality we can suppose that we have increasing families. Then

\[
\begin{align*}
 a_1 b_1 + a_2 b_2 + ... + a_n b_n &= a_1 b_1 + a_2 b_2 + ... + a_n b_n \\
a_1 b_1 + a_2 b_2 + ... + a_n b_n &\geq a_1 b_2 + a_2 b_3 + ... + a_n b_1 \\
a_1 b_1 + a_2 b_2 + ... + a_n b_n &\geq a_1 b_3 + a_2 b_4 + ... + a_n b_2 \\
& \vdots \\
a_1 b_1 + a_2 b_2 + ... + a_n b_n &\geq a_1 b_n + a_2 b_1 + ... + a_n b_{n-1}
\end{align*}
\]

Adding these inequalities we obtain the Chebyshev inequality.

### Applications

1. For any real numbers \( a, b \) we have \( (a + b)(a^2 + b^2)(a^4 + b^4) \leq 4(a^6 + b^6) \)

2. If \( a, b, c, \) respectively \( A, B, C, \) are the sides, respectively the angles in radians, of a triangle, then

\[ \frac{aA + bB + cC}{a + b + c} \geq \frac{\pi}{3} \]

3. Let \( n > 1 \) be an integer. If \( a_1, a_2, ..., a_n \) are positive real numbers then

\[ a_1^{a_1} a_2^{a_2} ... a_n^{a_n} \geq (a_1 a_2 ... a_n)^{\frac{a_1 + a_2 + ... + a_n}{n}} \]

4. If \( a, b, c \) are positive real numbers, prove that

\[ 2(a^3 + b^3 + c^3) \geq a^2(b + c) + b^2(c + a) + c^2(a + b) \]
5. Any sum of squares is nonnegative

This method is based on the following simple remarks:

1. If \( x \) is a real number, then \( x^2 \geq 0 \), with equality only for \( x = 0 \).
2. If \( x_1, x_2, \ldots, x_n \) are real numbers, then \( x_1^2 + x_2^2 + \ldots + x_n^2 \geq 0 \), with equality only for \( x_1 = x_2 = \ldots = x_n = 0 \).
3. If \( x_1, x_2, \ldots, x_n \) are real numbers, and \( a_1, a_2, \ldots, a_n \) are nonnegative real numbers, then \( a_1 x_1^2 + a_2 x_2^2 + \ldots + a_n x_n^2 \geq 0 \), with equality only for \( a_1 x_1 = a_2 x_2 = \ldots = a_n x_n = 0 \).

**Problem 9.9.** For any real numbers \( a, b, c \) the following inequalities hold:

a) \( ab + bc + ca \leq a^2 + b^2 + c^2 \)

b) \( abc(a + b + c) \leq a^4 + b^4 + c^4 \)

**Problem 9.10.** If \( a, b, c \) are nonnegative real numbers then

a) \((a^2 + b^2)c + (b^2 + c^2)a + (c^2 + a^2)b \geq 6abc\)

b) \(2(a^3 + b^3 + c^3) \geq (a + b)ab + (b + c)bc + (c + a)ca\)

**Problem 9.11.** Let \( a_1, a_2, \ldots, a_n \) be real numbers. Prove that

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \cos(a_i - a_j) \geq 0.
\]

6. Use monotonic functions

**Problem 9.12.** Let \( a, b, c, d \) be non-negative real numbers. Prove that

\[
\sqrt{a + b + c + d} \geq \sqrt{a + d} + \sqrt{b + a} + \sqrt{c + d} + a + b + c + d \geq 3\sqrt{a + b + c + d}
\]

When is the equality attained? Generalization.

**Problem 9.13.** If \( a, b, c \) are the sides of a triangle and \( n \) is a natural number, then

\[
\frac{2}{n+1} \leq \frac{a}{b+c+na} + \frac{b}{c+a+nb} + \frac{c}{a+b+nc} < \frac{3}{n+1}
\]

**Problem 9.14.** Prove that if \( x_k \geq a_k \geq 1 \), for \( k = 1, \ldots, n \), then

\[
\frac{(1 + a_1)(1 + a_2) \ldots (1 + a_n)}{2} \geq \frac{(a_1 + x_1)(a_2 + x_2) \ldots (a_n + x_n)}{a_1a_2 \ldots a_n + x_1x_2 \ldots x_n}
\]

**Problem 9.15.** If \( a_k \in [-1, 1] \) for \( k = 1, \ldots, n \), prove that

\[
2^n \geq (1 + a_1^2)(1 + a_2^2) \ldots (1 + a_n^2) + (1 - a_1^2)(1 - a_2^2) \ldots (1 - a_n^2)
\]

7. Study the behavior of a derivable function

**Problem 9.16.** Let \( x > -1 \) and \( a \) be real numbers.

a) If \( a \in (0, 1) \), then \( (1 + x)^a \leq 1 + ax \).

b) If \( a \notin (0, 1) \), then \( (1 + x)^a \geq 1 + ax \).

**Problem 9.17.** Prove that for any natural number \( n \)

\[
\left(1 + \frac{1}{n}\right)^{n+1/3} < e < \left(1 + \frac{1}{n}\right)^{n+1/2}
\]
8. Change the variables

8.1. Use symmetry, homogeneity

**Problem 9.18.** For any \( x, y, z > 0 \) prove that
\[
\frac{x}{\sqrt{(x+y)(x+z)}} + \frac{y}{\sqrt{(y+x)(y+z)}} + \frac{z}{\sqrt{(z+x)(z+y)}} \leq \frac{3}{2}.
\]

**Problem 9.19.** Let \( a, b \) be real numbers such that \( ab > 0 \). Prove that
\[
\sqrt[3]{a^2b^2(a+b)^2} \leq \frac{a^2 + 10ab + b^2}{12}.
\]
Determine when equality occurs. Hence, or otherwise, prove for all real numbers \( a, b \) that
\[
\sqrt[3]{a^2b^2(a+b)^2} \leq \frac{a^2 + ab + b^2}{3}.
\]
Determine the cases of equality.

8.2. Trigonometric substitutions

**Problem 9.20.** If \( n \in \mathbb{N} \), and \( |x| \leq 1 \), then \((1-x)^n + (1+x)^n \leq 2^n\).

**Problem 9.21.** If \( a, b \in [-1, 1] \), then \( \sqrt{1-a^2} + \sqrt{1-b^2} \leq \sqrt{4 - (a+b)^2} \).

9. Inequalities quadratic in one of the variables

**Problem 9.22.** For any real numbers \( a, b, c \) we have \( ab + bc + ca \leq a^2 + b^2 + c^2 \)

**Problem 9.23.** If \( a < b < c < d \), then \((a+b+c+d)^2 > 8(ac + bd)\).

10. Use convergent sequences

**Problem 9.24.** Show that for every positive integer \( n \),
\[
\left( \frac{2n-1}{e} \right)^{2n-1} \cdot 5 \cdot \cdots \cdot (2n-1) < \left( \frac{2n+1}{e} \right)^{2n+1}.
\]

**Problem 9.25.** Show that for any integer \( n \geq 1 \),
\[
\frac{1}{4n+2} < \ln 2 - \left( \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right) \leq \frac{1}{4n+1}
\]

**Problem 9.26.** Prove that for any natural number \( n \) the following inequality holds
\[
0 < e^{-\sum_{k=0}^{n} \frac{1}{k!}} < \frac{1}{n \cdot n!}.
\]
11. Variations

Problem 9.27. Let $a, b, c$ be positive real numbers. Then $a, b, c$ are the sides of a triangle if and only if
\[ 2(a^2b^2 + b^2c^2 + c^2a^2) > a^4 + b^4 + c^4. \]

Problem 9.28. Let $k < n$ be positive integers, and $a_1 < a_2 < \ldots < a_n$ be real numbers. Prove that
\[ \frac{a_1 + a_2 + \ldots + a_k}{k} < \frac{a_1 + a_2 + \ldots + a_n}{n} < \frac{a_{k+1} + a_{k+2} + \ldots + a_n}{n-k}. \]

Problem 9.29. Prove that for any $a, b, c$ sides of a triangle, we have
\[ (-a + b + c)(a-b+c)(a+b-c) \leq abc. \]

Solution: Changing the variables $-a + b + c = u$, $a - b + c = v$, $a + b - c = t$, the inequality can be arranged $(u + v)(v + t)(t + u) \geq 2\sqrt{uv}2\sqrt{vt}2\sqrt{tu}.$

Problem 9.30. For any $x, y, z > 0$ prove that
\[ \frac{x}{\sqrt{(x+y)(x+z)}} + \frac{y}{\sqrt{(y+x)(y+z)}} + \frac{z}{\sqrt{(z+x)(z+y)}} \leq \frac{3}{2}. \]

Solution: Changing the variables $x + y = a^2$, $y + z = b^2$, and $z + x = c^2$ the inequality can be arranged $(-a + b + c)(a - b + c)(a + b - c) \leq abc.$

Problem 9.31. Prove that $\ln x(x+1) > \ln(x+1)(x+2)$, for any $x > 1$.

Solution: Consequence of $\ln x \frac{x + 1}{x} > \ln x \frac{x + 2}{x + 1} > \ln x \frac{x + 2}{x + 1} > \ln x \frac{x + 2}{x + 1}.$

Problem 9.32. Show that $\log_2 3 + \log_3 4 + \log_4 5 + \log_5 6 > 5$.

Solution: The inequality of the means proves that $\log_2 3 + \log_3 4 + \log_4 5 + \log_5 6 \geq 4\sqrt[4]{\log_2 3 \log_3 4 \log_4 5 \log_5 6} = 4\sqrt[4]{1 + \log_2 3} > 4\sqrt[4]{1 + \frac{3}{2}} > 5$. The last inequality is a consequence of $2^7 = 128 > 125 = 5^3$.

Problem 9.33. Prove that for any positive integer $n$, we have $(n!)^2 > n^n$.

Solution: Using that $k(n + 1 - k) > n$, for any $k \in \{2, 3, \ldots, n - 1\}$, we have $(n!)^2 = n^2 \prod_{k=2}^{n-1} k(n + 1 - k) > n^n$.

Problem 9.34. For any positive integer $n$, $\sqrt{n} + \sqrt{n} + \sqrt{n} - \sqrt{n} \leq 2 \sqrt{n}$.

Solution: Consider the function $f(x) = \sqrt{x}$. The inequality writes $f(n + \sqrt{n}) - f(n) \leq f(n) - f(n - \sqrt{n})$. By the mean value theorem there are $c \in (n, n + \sqrt{n})$ such that $f(n + \sqrt{n}) - f(n) = \frac{1}{n} \sqrt{n} c^{1 \cdot \frac{1}{2}}$ and $d \in (n - \sqrt{n}, n)$ such that $f(n) - f(n - \sqrt{n}) = \frac{1}{n} \sqrt{n} d^{1 \cdot \frac{1}{2}}$. But $d < c$, $d^{\frac{1}{2}} < c^{\frac{1}{2}}$, which ends the proof.

Problem 9.35. Prove that $3 \cdot 5 \cdot 17 \cdots (1 + 2^n) < 2^{2n+1}$. 

Solution: Use \( (2^a - 1) \prod_{k=0}^{n} (1 + 2^k) = 2^{n+1} - 1. \)

Problem 9.36. Prove that for any integer \( n > 1, \)
\[
\frac{1}{n} - \frac{1}{2n+1} < \sum_{k=1}^{n} \frac{1}{k^2} < \frac{1}{n-1} - \frac{1}{2n}.
\]

Solution: Use \( \frac{1}{k} - \frac{1}{k+1} < \frac{1}{k^2} < \frac{1}{k-1} - \frac{1}{k}. \)

Problem 9.37. If \( n \geq 1 \) is an integer, then \( \sum_{k=1}^{n} \frac{1}{k^2} < \frac{7}{4}. \)

Solution: Using \( \frac{4}{a+b} \leq \frac{1}{a} + \frac{1}{b} \) we have
\[
\sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{2}{k(k+1) + k(k-1)} \leq 1 + \frac{1}{2} \sum_{k=2}^{n} \left( \frac{1}{k(k+1)} + \frac{1}{k(k-1)} \right) = 1 + \frac{1}{2} \left( 3 - \frac{1}{n+1} - \frac{1}{n} \right) < \frac{7}{4}.
\]

Problem 9.38. Let \( C \) be Euler’s constant and \( s_n = \sum_{k=1}^{n} \frac{1}{k}. \) Prove that \( C < s_n + s_m - s_{mn} < 1. \)

Solution: Let \( x_n = s_n - \ln n. \) It is known that \( x_n \) is decreasing to \( C, \) so \( x_n > C, \) for any \( n. \) Then \( s_n + s_m - s_{mn} = x_n + (x_m - x_{mn}) > x_n > C. \)

Also from \( s_{kn} - s_{(k-1)n} = \sum_{p=1}^{n} \frac{1}{(k-1)n + p} \geq \sum_{p=1}^{n} \frac{1}{kn} = \frac{1}{k} \) we have \( s_n + s_m - s_{mn} = s_m + \sum_{k=1}^{m-1} \frac{1}{kn}, \) so \( s_m - \sum_{k=1}^{m-1} \frac{1}{k+1} = 1. \)

Problem 9.39. Let \( a \) be a fixed real number. If \( a \in (0, 1), \) then \( (1+x)^a \leq 1 + ax \) for any \( x > -1, \) and if \( a \notin (0, 1), \) then \( (1+x)^a \geq 1 + ax. \)

Solution: Study the behavior of the function \( f : (-1, \infty) \to \mathbb{R}, f(x) = (1+x)^a - 1 - ax. \)

Problem 9.40. Let \( a, b, c, d \) be non-negative real numbers. Prove that
\[
\sqrt{a+b+c} + \sqrt{b+c+d} + \sqrt{c+d+a} + \sqrt{d+a+b} \geq 3\sqrt{a+b+c+d}
\]
When is the equality attained? Generalization.

Solution: Consider the functions \( f(a) = \sqrt{a+b+c} + \sqrt{b+c+d} + \sqrt{c+d+a} + \sqrt{d+a+b} - 3\sqrt{a+b+c+d} \) and \( g(b) = \sqrt{b+c} + \sqrt{c+d} + \sqrt{d+a} + \sqrt{a+b} - 2\sqrt{b+c+d}. \)

Since \( g'(b) = \left( \frac{1}{2\sqrt{b+c}} - \frac{1}{2\sqrt{b+c+d}} \right) + \left( \frac{1}{2\sqrt{b+d}} - \frac{1}{2\sqrt{b+c+d}} \right) \geq 0, \) the function \( g \) is increasing and \( g(b) \geq g(0) = \sqrt{a} + \sqrt{c} + \sqrt{d} \geq 0. \) Also
\[
f'(a) = \left( \frac{1}{2\sqrt{a+b+c}} - \frac{1}{2\sqrt{a+b+c+d}} \right) + \left( \frac{1}{2\sqrt{a+c+d}} - \frac{1}{2\sqrt{a+b+c+d}} \right)
\]
Problem 9.41. Let \( a_1, a_2, \ldots, a_n \) be given. Find the least value of \( x_1^2 + x_2^2 + \ldots + x_n^2 \) given that \( a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 1 \).

Solution: One can solve the problem using Lagrange’s multipliers, but we prefer a more elementary approach. The Cauchy-Schwartz inequality gives

\[
(x_1^2 + x_2^2 + \ldots + x_n^2)(a_1^2 + a_2^2 + \ldots + a_n^2) \geq (a_1 x_1 + a_2 x_2 + \ldots + a_n x_n)^2 = 1
\]

The equality holds when \( \frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n} \) does not depend on \( k \). Therefore the least value of \( x_1^2 + x_2^2 + \ldots + x_n^2 \) is \( 1/(a_1^2 + a_2^2 + \ldots + a_n^2) \).

Problem 9.42. Prove that for all positive integers \( n \)

\[
\begin{align*}
1. & \quad \frac{2n}{3n+1} \leq \sum_{k=n+1}^{2n} \frac{1}{k} \\
2. & \quad \sum_{k=n+1}^{2n} \frac{1}{k} \leq \frac{3n+1}{4n+4}
\end{align*}
\]

Solution: 1. We proceed by induction. For \( n = 1 \) the inequality is trivial. We suppose the inequality satisfied for \( n \). Since \( \sum_{k=n+1}^{2n+2} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k} + \frac{1}{2n+1} - \frac{1}{2n+2} \), it suffices to prove \( \frac{1}{2n+1} - \frac{1}{2n+2} \geq \frac{2n + 2}{3n+4} - \frac{2n}{3n+1} \). Inequality equivalent with \( 9n^2 + 15n + 4 = (3n+1)(3n+4) > 2(2n+1)(2n+2) = 8n^2 + 12n + 4. \)

2. Similar with 1. It suffices to prove the inequality \( \frac{1}{2n+1} - \frac{1}{2n+2} \leq \frac{3n+4}{4n+8} \) which is equivalent with \( \frac{1}{(2n+1)(2n+2)} \leq \frac{3n+4}{4(n+1)(n+2)} \).

Problem 9.43. Let \( a, b \) be real numbers such that \( ab > 0 \). Prove that

\[
\sqrt[3]{\frac{a^2 b^2 (a + b)^2}{4}} \leq \frac{a^2 + 10ab + b^2}{12}.
\]

Determine when equality occurs. Hence, or otherwise, prove for all real numbers \( a, b \) that

\[
\sqrt[3]{\frac{a^2 b^2 (a + b)^2}{4}} \leq \frac{a^2 + ab + b^2}{3}.
\]

Determine the cases of equality.

Solution: Without any loss of generality one can suppose \( a > 0, b > 0 \). With the notations \( S = a + b \) and \( P = ab \) the inequality can be arranged as \( \sqrt[3]{\frac{S^2 P^2}{4}} \leq \frac{S^2 + 8P}{12} \). Since \( S^2 \geq 4P \), there exist \( t \geq 1 \) such that \( S^2 = 4Pt \). The inequality is then equivalent to \( \sqrt[3]{t \cdot 1} \leq \frac{t + 1 + 1}{3} \) which is the inequality of means, with equality for \( t = 1 \), or \( a = b \).
For \( ab > 0 \) the second inequality is a consequence of the first one and of the inequality \( \frac{a^2 + 10ab + b^2}{12} \leq \frac{a^2 + ab + b^2}{3} \). If \( ab = 0 \) then at least one of \( a \) and \( b \) is zero and the inequality is trivial.

For \( ab < 0 \), we can suppose without any restriction of the generality that \( a > 0 \) and \( b < 0 \). Let \( c = -b > 0 \) and \( t = \frac{a}{c} > 0 \). The inequality is equivalent with \( 27t^2(t - 1)^2 \leq 4(t^2 - t + 1)^3 \), and a multiplication by \( \frac{1}{t^3} \) gives \( 27(t + 1/t - 2) \leq 4(t + 1/t - 1)^3 \). Let \( x = t + 1/t - 1 \geq 1 \). It suffices to show that \( 4x^3 \geq 27(x - 1) \).

But \( 4x^3 - 27x + 27 = (2x - 3)^2(x + 3) \geq 0 \). We have equality when \( x = \frac{3}{2} \), which corresponds either to \( t = 2 \), i.e. \( a = -2b \), or \( t = \frac{1}{2} \), i.e. \( b = -2a \).

**Problem 9.44.** Prove or disprove: If \( x \) and \( y \) are real numbers with \( y \geq 0 \) and \( y(y + 1) \leq (x + 1)^2 \), then \( y(y - 1) \leq x^2 \). \([P1988]\)

**Solution:** The statement is true. The inequalities \( y \geq 0 \) and \( y(y + 1) \leq (x + 1)^2 \) are equivalent with \( 0 \leq y \leq \sqrt{(x + 1)^2 + \frac{1}{4}} - \frac{1}{2} \). Thus \(|y - \frac{1}{2}| \leq \max \left( \frac{1}{2}, |\sqrt{(x + 1)^2 + \frac{1}{4}} - \frac{1}{2}| \right) \). But \( \frac{1}{2} \leq \sqrt{x^2 + \frac{1}{4}} \) and also \(|\sqrt{(x + 1)^2 + \frac{1}{4}} - \frac{1}{2}| \leq \sqrt{x^2 + \frac{1}{4}} \), hence \(|y - \frac{1}{2}| \leq \sqrt{x^2 + \frac{1}{4}} \) which is an equivalent form of \( y(y - 1) \leq x^2 \).

**Problem 9.45.** Show that for every positive integer \( n \),

\[
\left( \frac{2n - 1}{e} \right)^{2n-1} e^2 < 1 \cdot 3 \cdot 5 \cdots (2n - 1) \leq \left( \frac{2n + 1}{e} \right)^{2n+1}.
\]

\([P1996]\)

**Solution:** We will use the inequality \( x > \ln(1 + x) > \frac{x}{1 + x} \), which holds for any positive \( x \). Taking the logarithm, the left side of the inequality to prove becomes

\[
\frac{2n - 1}{2} (\ln(2n - 1) - 1) < \ln(2n + 1) + \ln(2n - 1).
\]

We prove this by induction. The verification for \( n = 1 \) is trivial. We suppose the inequality satisfied for \( n = k \). Then

\[
\ln(2k + 1) + \ln(2k - 1) > \frac{2k - 1}{2} (\ln(2k - 1) - 1) + \ln(2k + 1)
\]

It suffices to show that

\[
\frac{2k - 1}{2} (\ln(2k - 1) - 1) + \ln(2k + 1) > \frac{2k + 1}{2} (\ln(2k + 1) - 1)
\]

which is equivalent with

\[
\frac{2}{2k - 1} > \ln \left( 1 + \frac{2}{2k + 1} \right)
\]

The right side of the inequality is equivalent with

\[
\frac{2n + 1}{2} (\ln(2n + 1) - 1) > \ln(2n + 1) + \ln(2n - 1)
\]

We proceed by induction. The verification for \( n = 1 \) is trivial. Let us suppose that this inequality holds for \( n = k \). Then

\[
\ln(2k + 1) + \ln(2k - 1) > \frac{2k + 1}{2} (\ln(2k + 1) - 1) + \ln(2k + 1)
\]

It suffices to show that
PROBLEM 9.46. Prove that for any \( x \geq 1 \) and for any positive integer \( n \) the following inequalities holds

\[
\begin{align*}
\text{(i)} & \quad \sqrt{x + 1} + \sqrt{x + 2} + \ldots + \sqrt{x + n} < x + 1 \\
\text{(ii)} & \quad \sqrt{x + 1} + \sqrt{x + 2} + \ldots + \sqrt{x + n} < \sqrt{2(x + 1)}
\end{align*}
\]

Solution: (Alex Kumjian) Consider the sequence \( (a_n) \), defined by \( a_1 = (x + 1)^2 \), \( a_{n+1} = (a_n - x - n - 1)^2 \) for \( n \geq 1 \). An easy induction shows that \( a_n > 2(n - 1 + x) \). Also another induction based on \( x + n - 1 + \sqrt{a_{n+1}} = a_n \) proves that

\[
\sqrt{x + 1} + \sqrt{x + 2} + \ldots + \sqrt{x + n} < \sqrt{x + (x + 2)}
\]

Second solution (Chris Herald) Using successively the inequality \( \frac{a + 1}{2} \geq \sqrt{a} \) we get

\[
\sqrt{x + 1} + \sqrt{x + 2} + \ldots + \sqrt{x + n} \leq \sum_{k=1}^{n+1} \frac{x + k}{2k} < \sum_{k=1}^{\infty} \frac{x + k}{2k} = x + 2
\]

PROBLEM 9.47. Show that for any integer \( n \geq 1 \),

\[
\frac{1}{4n + 2} < \ln 2 - \frac{1}{n + 1} + \frac{1}{n + 2} + \ldots + \frac{1}{2n} < \frac{1}{4n + 1}
\]

Solution: Consider the sequence \( a_n = \frac{1}{n + 1} + \frac{1}{n + 2} + \ldots + \frac{1}{2n} + \frac{1}{4n + 1} \). Since \( a_{n+1} - a_n = \frac{1}{(2n + 1)(2n + 2)} - \frac{1}{(2n + 1)(2n + 3)} > 0 \) the sequence is increasing. But \( \lim a_n = \ln 2 \), so \( a_n < \ln 2 \), which proves the left inequality.

Let \( b_n = \frac{1}{n + 1} + \frac{1}{n + 2} + \ldots + \frac{1}{2n} + \frac{1}{4n + 1} \). From

\[
b_{n+1} - b_n = -\frac{3}{(2n + 1)(2n + 2)(4n + 1)(4n + 5)} < 0
\]

we see that the sequence \( (b_n) \) is decreasing. But \( \lim b_n = \ln 2 \), so \( b_n > \ln 2 \), which proves the right inequality.
Problem 9.48. Let \(a_1, a_2, \ldots, a_n\) be real numbers. Prove that
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \cos(a_i - a_j) \geq 0.
\]
Solution: \[
\sum_{i=1}^{n} \sum_{j=1}^{n} (\cos a_i \cos a_j + \sin a_i \sin a_j) = \left( \sum_{i=1}^{n} \cos a_i \right)^2 + \left( \sum_{i=1}^{n} \sin a_i \right)^2 \geq 0.
\]

Problem 9.49. If \(a < b < c < d\), then \((a + b + c + d)^2 > 8(ac + bd)\).
Solution: Suffice to prove that \(f(d) = d^2 + 2d(a - 3b + c) + (a + b + c)^2 - 8ac > 0\), for any \(d\), knowing that \(a < b < c\). And this is a consequence of \(\Delta = 8(b-a)(b-c) < 0\).

Problem 9.50. Prove that for any \(x, y \in [0, 1]\) we have the inequality
\[
\sqrt{x+y} \geq x\sqrt{1-y} + y\sqrt{1-x}
\]
Solution: Using Cauchy-Schwarz inequality
\[
\sqrt{x^2} + \sqrt{y^2} \leq \sqrt{(x+y+0)(-x+y+0)}
\]
To end the proof it suffices to remark that \(x + y - 2xy \leq 1\). The equality takes place for \(x = 0, y = 1\) and \(x = 1, y = 0\).

Problem 9.51. Let \(a, b, c\) be positive numbers such that \(a \geq c, b \geq c\). Prove that
\[
\sqrt{a-c} + \sqrt{b-c} \leq \sqrt{\frac{ab}{c}}.
\]
Solution: Use Cauchy-Schwarz
\[
\sqrt{c(a-c)} + \sqrt{(b-c)c} \leq \sqrt{(c+b-c)(a-c+c)}.
\]
CHAPTER 10

Polynomials

Let \( n \) be a non-negative integer and \( a_0, a_1, \ldots, a_n \) elements of a ring \( A \) (which can be \( \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{Z}_m \)), with \( a_n \neq 0 \). The polynomial \( P \) in the variable \( x \), of degree \( n \), and coefficients \( a_0, a_1, \ldots, a_n \) is the expression

\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0
\]

- The coefficient \( a_n \) is called the leading coefficient. If it is equal to 1, we say \( P \) is a monic polynomial.
- By definition the polynomial 0 has degree \(-\infty\).
- Two polynomials are equal if their degrees are the same and the coefficients are equal term for term.
- The set of all polynomials with coefficients in the ring \( A \) is denoted \( A[x] \).
- The divisibility and the greatest common divisor for polynomials are defined in the same way like for integers.
- An element \( a \) is called a root, or zero of the polynomial \( F \), if \( F(a) = 0 \). We say \( a \) is a zero of multiplicity \( q \), if there is a polynomial \( Q \) such that \( Q(a) \neq 0 \) and \( F(x) = (x-a)^q Q(x) \).

**Bézout’s Theorem**

If \( F \) is a polynomial over \( \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z} \) or \( \mathbb{Z}_p \), with \( p \) prime, \( x - a \) is a divisor of \( F \) if and only if \( a \) is a zero of \( F \).

**The Fundamental Theorem of Algebra**

Every polynomial with complex coefficients of degree at least 1 has a complex root.

**The Factorization Theorem**

Every polynomial \( P(x) \) with complex coefficients and degree at least 1 can be written uniquely as a product

\[
P(x) = \alpha(x - a_1)(x - a_2)\ldots(x - a_n)
\]

where \( \alpha \) and \( a_1, a_2, \ldots, a_n \) are complex numbers, not necessarily distinct.

Every polynomial \( P(x) \) with real coefficients and degree at least 1 can be written uniquely as a product

\[
P(x) = \alpha(x - a_1)(x - a_2)\ldots(x - a_r)(x^2 + b_1 x + c_1)\ldots(x^2 + b_s x + c_s)
\]

where \( \alpha \) and \( a_i, b_i, c_i \) are real numbers, not necessarily distinct, and \( b_i^2 - 4c_i < 0 \), for \( i = 1, 2, \ldots, s \).
1. Division with remainder

The division with remainder theorem

If \( F \) and \( G \) are polynomials over a field (for example \( \mathbb{C}, \mathbb{R}, \mathbb{Q} \) or \( \mathbb{Z}_p \) for \( p \) prime), then there exist unique polynomials \( Q \), called quotient, and \( R \), called remainder, over the same field, with \( \deg R < \deg G \), such that

\[
F = QG + R.
\]

In the particular case of \( G(x) = x - a \), the remainder is \( F(a) \).

As in the case of integers, Euclid’s algorithm can be used to find the g.c.d. of two polynomials. Furthermore, if \( F \) and \( G \) are polynomials, there are polynomials \( U \) and \( V \) such that \( \text{g.c.d}(F, G) = UF + VG \).

Problem 10.1. Find the polynomials \( U \) and \( V \) such that \((x^7 - 1)U(x) + (x^4 - 1)V(x) = x - 1\).

Problem 10.2. Let \( m, n \) be positive integers. The division of \( m \) by \( n \) gives the remainder \( r \). Prove that the division of \( x^m - 1 \) by \( x^n - 1 \) gives \( x^r - 1 \).

Problem 10.3. Determine \( \text{g.c.d}(x^m - 1, x^n - 1) \).

Problem 10.4. Prove that the fraction \( \frac{n^3 + 2n}{n^2 + 3n^2 + 1} \) is irreducible for every integer \( n \).

Problem 10.5. Let \( P(x) \) be a polynomial such that \( P(0) = 3, P'(0) = 7 \), and \( P(1) = 8 \). Compute the remainder (polynomial) you get when you divide \( P(x) \) by \( x^3 - x^2 \).

Solution: The remainder is of the second degree, so there are real numbers \( a, b, c \) such that \( P(x) = (x^3 - x^2)Q(x) + ax^2 + bx + c \). The three conditions on \( P(x) \) translate in the system

\[
\begin{align*}
c &= 3 \\
b &= 7 \\
a + b + c &= 8
\end{align*}
\]

so the remainder is \(-2x^2 + 7x + 3\).

Problem 10.6. Let \( P \) be a polynomial with real coefficients such that \( P(\sin t) = P(\cos t) \) for all \( t \in \mathbb{R} \). Prove that there is a polynomial \( Q \) such that \( P(x) = Q(x^4 - x^2) \).

Solution: There is a polynomial \( P_1 \) and real numbers \( a, b, c, d \) such that

\[
P(x) = (x^4 - x^2)P_1(x) + ax^3 + bx^2 + cx + d.
\]

By hypothesis

\[
-\sin^2 t \cos^2 t P_1(\sin^2 t) + a \sin^3 t + b \sin^2 t + c \sin t = -\sin^2 t \cos^2 t P_1(\cos^2 t) + a \cos^3 t + b \cos^2 t + c \cos t
\]

For \( t = 0 \) we get \( a + b + c = 0 \), and \( t = \pi \) gives \( -a + b - c = 0 \), so \( b = 0 \) and \( a = -c \). The polynomial \( P_1 \) is satifying then \( \sin t \cos t P_1(\sin t) + a \cos t = \sin t \cos t P_1(\cos t) + a \sin t \) which for \( t = 0 \) gives \( a = 0 \), so \( P_1(\sin t) = P_1(\cos t) \).

Repeating the process proves the statement.

Problem 10.7. Let \( m, n \) be positive integers. Prove that if \( (m, n) = 1 \) then the polynomials of complex variable \( P(z) = z^n - 1 \) and \( Q(z) = z^m - 1 \) have only one common root.
Then \( z \in \mathbb{C} \) be a common root and \( a, b \) integers such that \( an + bm = 1 \). Then \( z = z^{an+bm} = (z^n)^a(z^m)^b = 1 \).

**Problem 10.8.** Prove that every polynomial with complex coefficients has a nonzero polynomial multiple whose exponents are all divisible by 100.

**Solution:** Let \( P(x) \) be a polynomial of degree \( n \). For any \( k = 1, 2, \ldots, n-1, n \), there are polynomials \( Q_k, R_k \), with deg \( R_k \leq n-1 \), such that
\[
x^{100k} = Q_k(x)P(x) + R_k(x)
\]
The \( n \) polynomials \( R_1(x), R_2(x), \ldots, R_n(x) \) are linearly dependent in the vector space of polynomials of degree smaller than \( n-1 \). Then there are complex numbers \( a_1, a_2, \ldots, a_n \), not all zero, such that
\[
a_1R_1(x) + a_2R_2(x) + \ldots + a_nR_n(x) = 0.
\]
Consequently,
\[
a_1x^{100} + a_2x^{200} + \ldots + a_nx^{100n} = (a_1Q_1(x) + a_2Q_2(x) + \ldots + a_nQ_n(x))P(x)
\]

**Problem 10.9.** Find the remainder when \( (x+a)^n \) is divided by \( (x+b)^m \).

**Solution:** Denote \( P(x) = (x+a)^n \) and let \( R(x) \), respectively \( Q(x) \), be the remainder, respectively the quotient of \( P(x) \) when divided by \( (x+b)^m \). The degree of \( R(x) \) is smaller than \( m-1 \), so there are \( a_{m-1}, a_{m-2}, \ldots, a_1, a_0 \) such that
\[
P(x) = Q(x)(x+b)^m + a_{m-1}(x+b)^{m-1} + a_{m-2}(x+b)^{m-2} + \ldots + a_1(x+b) + a_0
\]
For \( k = 0, 1, 2, \ldots, m-1 \) evaluating the \( k \)-th derivative of this relation in \( x = -b \) we get \( P^{(k)}(-b) = k!a_k \). Therefore \( a_k = \binom{n}{k}(a-b)^{n-k} \) and
\[
Q(x) = \sum_{k=0}^{m-1} \binom{n}{k}(a-b)^{n-k}(x+b)^k.
\]

**Problem 10.10.** If \( P \) is a polynomial with real coefficients, prove that \( P(P(P(P(x)))) - x \) is divisible by \( P(x) - x \).

**Problem 10.11.** Determine the remainder of the polynomial \( F \) at the division by \( (x-a)(x-b) \).

**Problem 10.12.** Let \( n \) be a positive integer. Prove that a monic polynomial \( P \) of degree \( n \) is divisible by the sum of all its derivatives if and only if there is \( a \in \mathbb{C} \) such that
\[
P(x) = (x-a)(x-a+n)^{n-1}
\]

**Solution:** Assume that \( P \) is divisible by \( P' + P^{(2)} + \ldots + P^{(n)} \). Then there exist complex numbers \( a, c \) such that
\[
P(x) = c(x-a) \left( P'(x) + P^{(2)}(x) + \ldots + P^{(n)}(x) \right)
\]
Identifying the leading coefficients we obtain \( c = \frac{1}{n} \). We derivate the equality 1.1 and get
\[
P'(x) = \frac{1}{n} \left( P'(x) + P^{(2)}(x) + \ldots + P^{(n)}(x) \right) + \frac{x-a}{n} \left( P^{(2)}(x) + \ldots + P^{(n)}(x) \right)
\]
and from here

\[ P^{(2)}(x) + \ldots + P^{(n)}(x) = \frac{(n-1)P'(x)}{x - a + 1} \]

We substitute back in the equation 1.1 and we obtain

\[ \frac{P'(x)}{P(x)} = \frac{1}{x - a} + \frac{n - 1}{x - a + n} \]

We solve this differential equation and obtain

\[ P(x) = (x - a)(x - a + n)^{n-1} \]

Conversely, consider the polynomial

\[ P(x) = (x - a)(x - a + n)^{n-1} = (x - a + n)^n - n(x - a + n)^{n-1} \]

We derivate successively and by telescopic summation we have

\[ P'(x) + P^{(2)}(x) + \ldots + P^{(n)}(x) = n(x - a + n)^{n-1} \]

which divides \( P(x) \).

**Problem 10.13.** Let \( P \) be a polynomial of complex coefficients. Prove that \((P \circ P \circ \ldots \circ P)(x) - x\) is divisible by \( P(x) - x \).

### 2. Relations between roots and coefficients

**Theorem (Viète)** If \( x_1, x_2, \ldots, x_n \) are the roots of the polynomial \( P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) then we have

\[
\begin{align*}
S_1 &= x_1 + x_2 + \ldots + x_n = -\frac{a_{n-1}}{a_n} \\
S_2 &= \sum_{i<j} x_ix_j = -\frac{a_{n-2}}{a_n} \\
S_3 &= \sum_{i<j<k} x_ix_jx_k = -\frac{a_{n-3}}{a_n} \\
\vdots & \quad \vdots \\
S_n &= x_1x_2\ldots x_n = (-1)^n \frac{a_0}{a_n}
\end{align*}
\]

**Problem 10.14.** Let \( x_1, x_2, x_3 \) be the roots of the polynomial \( P(x) = x^3 + x^2 + 3x + 1 \). Determine

1. \( S_1 = x_1 + x_2 + x_3, S_2 = x_1x_2 + x_1x_3 + x_2x_3, S_3 = x_1x_2x_3 \).
2. \( x_1^2 + x_2^2 + x_3^2 \)
3. \( x_1^3 + x_2^3 + x_3^3 \)
4. \( x_1^4 + x_2^4 + x_3^4 \)
5. \( \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{1+x_1} + \frac{1}{1+x_2} + \frac{1}{1+x_3} \)
6. \( \frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} \)

**Problem 10.15.** Let \( a, b, c \) be real numbers, and \( S_1 = a+b+c, S_2 = ab+bc+ca, S_3 = abc. \) Prove that \( a, b, c \) are all three positive numbers if and only if \( S_1, S_2 \) and \( S_3 \) are all three positive numbers. Generalization.
Let us denote \( Q(a,b,c) = \{a_1, a_2, a_3, a_4\} = \{b_1, b_2, b_3, b_4\} \).

**Problem 10.16.** Let \( a_1, a_2, a_3, a_4 \), and \( b_1, b_2, b_3, b_4 \) be eight real numbers such that \( \sum_{i=1}^{4} a_i^k = \sum_{i=1}^{4} b_i^k \), for any \( k = 1, 2, 3, 4 \). Prove that \( \{a_1, a_2, a_3, a_4\} = \{b_1, b_2, b_3, b_4\} \).

**Problem 10.17.** Let \( a, b, c \) be real numbers such that \( a + b + c = 0 \). Prove that \( \frac{a^5 + b^5 + c^5}{5} = \left( \frac{a^3 + b^3 + c^3}{3} \right) \left( \frac{a^2 + b^2 + c^2}{2} \right) \).

**Problem 10.18.** Let \( p, q, r \) be complex numbers and \( a, b, c \) the distinct complex zeros of the polynomial \( P(x) = x^3 + px^2 + qx + r \). Express in terms of \( p, q, r \) the expression
\[
E = \frac{a^4}{(a-b)(a-c)} + \frac{b^4}{(b-a)(b-c)} + \frac{c^4}{(c-a)(c-b)}
\]

**Solution:** We have
\[
E = \frac{a^4(b-c) - b^4(a-c) + c^4(a-b)}{(a-b)(a-c)(b-c)}
\]
Let us denote \( Q(a, b, c) = a^4(b-c) - b^4(a-c) + c^4(a-b) \). Since \( Q(a, a, c) = 0 \) we note that \( Q(a, b, c) \) should have the factor \( a - b \). Similarly we note that \( Q(a, b, c) \) should also have the factors \( b - c \) and \( a - c \). We group then conveniently and we can factor
\[
Q(a, b, c) = (a - b)(a - c)(b - c)(a^2 + b^2 + c^2 + ab + ac + bc).
\]
Hence \( E = a^2 + b^2 + c^2 + ab + ac + bc = p^2 - q \).

**Problem 10.19.** Find the real number knowing that the equation \( x^3 - 3x^2 + ax - 1 = 0 \) has three positive solutions. Solve the equation.

**Problem 10.20.** Show that the polynomial equation with real coefficients \( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_3 x^3 + x^2 + x + 1 = 0 \) has fewer than \( n \) real roots.

**Problem 10.21.** Solve the equation \( x^4 + ax^3 + bx^2 - 4x + 1 = 0 \) knowing that it has 4 positive roots.

**Problem 10.22.** Solve the system. Generalization.
\[
\begin{align*}
8x - 4y + 2z - t &= 16 \\
27x - 9y + 3z - t &= 81 \\
64x - 16y + 4z - t &= 256.
\end{align*}
\]

**Problem 10.23.** Let \( P(x) = x^3 + ax^2 + bx + c \) be a polynomial with real coefficients, \( c \neq 0 \), and \( x_1, x_2, x_3 \) the roots of \( P(x) \).
\( a) \) Determine the polynomial \( Q(x) \) whose roots are \( \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3} \).
\( b) \) Find a necessary and sufficient condition for \( P \) and \( Q \) to have a common root. Determine \( x_1, x_2, x_3 \) in this case.

**Problem 10.24.** Solve the system of equations
\[
\begin{align*}
x + y + z &= 2 \\
x^2 + y^2 + z^2 &= 6 \\
x^6 + y^6 + z^6 &= 66
\end{align*}
\]
10. POLYNOMIALS

**Problem 10.25.** Prove that the polynomials $P(x) = x^3 - x^2 + \frac{3}{2}x + a$ and $Q(x) = x^3 - 6x^2 + \frac{27}{2}x + b$ cannot have all the roots real.

**Solution:** Use Viète relations or observe that as functions they’re both strictly increasing over the reals (positive derivative).

**Problem 10.26.** Let $x_1, x_2, \ldots, x_n$ be the roots of the polynomial $P_n(x) = x^n + x^{n-1} + \ldots + x + 1$. Evaluate $S_n = \sum_{i=1}^{n} \frac{1}{x_k - a}$ where $a \neq -1$.

**Problem 10.27.** Let $a, b, c$ be real numbers such that $a + b + c = 1$. Prove that $10|a^3 + b^3 + c^3 - 1| \leq 9|a^5 + b^5 + c^5 - 1|$.

**Solution:** Consider the polynomial $P(x) = x^3 - x^2 + px - q$ which has the roots $a, b, c$. Then $ab + bc + ca = p$ and $abc = q$. With the notation $S_n = a^n + b^n + c^n$, we have $S_{n+3} - S_{n+2} + pS_{n+1} - qS_n = 0$. We evaluate directly $S_2 = (a + b + c)^2 - 2(ab + bc + ca) = 1 - 2p$, then by recurrence $S_3 = 1 - 3p + 3q$, $S_4 = 2p^2 - 4p + 4q + 1$, and $S_5 = 5p^2 - 5p + 5q - 5pq + 1$. The inequality to prove becomes $\frac{2}{3} \leq |p - 1|$, and this is a consequence of $p = ab + bc + ca \leq \frac{1}{3}(a + b + c)^2 = \frac{1}{3}$.

**Problem 10.28.** Find the real values of $m$ for which the system $u + v + w = 0$, $u^3 + v^3 + w^3 = 3m$, $u^7 + v^7 + w^7 = 63m$ has real solutions.

**Solution:** Let $P(x) = x^3 + bx - c$ be the polynomial with roots $u, v, w$. Then $S_n = u^n + v^n + w^n$ satisfies the recurrence $S_{n+3} + bS_{n+1} - cS_n = 0$. We have $S_2 = -2b$, $S_3 = 3c$, $S_4 = 2b^2$, $S_5 = -5bc$, $S_7 = 7b^7c$, hence $c = m$ and $b^2c = 9m$.

We distinguish the cases:

1. $b = 3$. Since $P'(x) > 0$, $P$ is an increasing function and it has only one real root.

2. $b = -3$. Since $P'(x) = 3(x^2 - 1)$, $P$ is increasing on $(-\infty, -1)$, decreasing on $(-1, 1)$, and increasing on $(1, \infty)$. Then $P$ has 3 real roots if and only if $P(-1) = 2 - m > 0$ and $P(1) = -2 - m < 0$, or $m \in (-2, 2)$.

**Problem 10.29.** Find the real parameter $a$ and solve the equation $x^3 - 6x^2 + ax + a = 0$, knowing that the roots $x_1, x_2, x_3$ satisfy $(x_1-1)^3 + (x_2-2)^3 + (x_3-3)^3 = 0$.

**Solution:** Since $(x_1 - 1) + (x_2 - 2) + (x_3 - 3) = 0$, the identity $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$ proves $3(x_1 - 1)(x_2 - 2)(x_3 - 3) = (x_1 - 1)^3 + (x_2 - 2)^3 + (x_3 - 3)^3 = 0$. We distinguish the cases:

1. $x_1 = 1$. Then $a = \frac{2}{3}$ and $x_2, x_3$ are the roots of $x^2 - 5x - \frac{2}{3} = 0$.
2. $x_2 = 2$. Then $a = \frac{16}{3}$ and $x_1, x_3$ are the roots of $x^2 - 4x - \frac{8}{3} = 0$.
3. $x_3 = 3$. Then $a = \frac{27}{4}$ and $x_1, x_2$ are the roots of $x^2 - 3x - \frac{9}{4} = 0$.

**Problem 10.30.** Solve the equation $x^4 + 8x^3 + ax^2 + bx + 16 = 0$ knowing it has 4 negative real roots.

**Solution:** Let $x_1, x_2, x_3, x_4$ be the negative roots of the equation. Then the numbers $y_i = x_i, i = 1, 4$ are positive and $\frac{y_1 + y_2 + y_3 + y_4}{4} = 2 = \sqrt[4]{y_1y_2y_3y_4}$. But
2. RELATIONS BETWEEN ROOTS AND COEFFICIENTS

the equality in the inequality between the arithmetic and geometric means takes place only for \( y_1 = y_2 = y_3 = y_4 = 2 \). In consequence \( x_1 = x_2 = x_3 = x_4 \), and \( a = 24 \), and \( b = 32 \).

**Problem 10.31.** Let \( P(x) = 2x^3 - 2x^2 + 3x + a \), where \( a \) is a real number, and \( x_1, x_2, x_3 \) the roots of \( P \).

a) Determine \( x_1^2 + x_2^2 + x_3^2 \), \( \frac{x_1}{1 + x_1} + \frac{x_2}{1 + x_2} + \frac{x_3}{1 + x_3} \).

b) How many of the roots \( x_1, x_2, x_3 \) are real numbers?

**Problem 10.32.** If \( x_1, x_2, ... , x_n \) are the roots of the equation \( x^n + 2x^{n-1} + ... + nx + (n + 1) = 0 \), evaluate \( \sum_{i=1}^{n} \frac{x_i^{n+1} - 1}{x_i - 1} \).

**Solution:** The identity \( \sum_{k=0}^{n+1} y^k = \frac{y^{n+2} - 1}{y - 1} \) by derivation gives \( \sum_{k=0}^{n} (k + 1)y^k = \frac{(n + 1)y^{n+2} - (n + 2)y^{n+1} + 1}{(y - 1)^2} \). Taking \( y = \frac{1}{x_i} \) we obtain \( \frac{x_i^{n+2} - (n + 2)x_i + (n + 1)}{(x_i - 1)^2} = \sum_{k=0}^{n} (k + 1)x_i^{n-k} = 0 \).

Therefore \( \frac{x_i^{n+1} - 1}{x_i - 1} = \frac{n + 2 - \frac{n+1}{x_i}}{x_i - 1} = \frac{n + 1}{x_i} \), and \( \sum_{i=1}^{n} \frac{x_i^{n+1} - 1}{x_i - 1} = (n + 1) \sum_{i=1}^{n} \frac{1}{x_i} = -n \).

**Problem 10.33.** Determine a necessary and sufficient condition for the system

\[
\begin{align*}
x + y + z &= A \\
x^2 + y^2 + z^2 &= B \\
x^3 + y^3 + z^3 &= C
\end{align*}
\]

to have a solution with at least one of \( x, y, z \) equal to 0.

**Solution:** The numbers \( x, y, z \) are the zeros of the polynomial \( P(t) = t^3 - a_1t^2 + a_2t + a_3 \), where

\[
\begin{align*}
a_1 &= x + y + z \\
a_2 &= xy + yz + zx \\
a_3 &= xyz
\end{align*}
\]

One of the numbers \( x, y, z \) is equal to 0 if and only if \( a_3 = 0 \). But \( a_2 = (A^2 - B)/2 \) and the equality

\[
x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)
\]
yields \( xyz = \frac{1}{3} \left( C - \frac{3AB}{2} + \frac{A^3}{2} \right) \). Therefore the condition is \( 2C - 3AB + A^3 = 0 \).

**Problem 10.34.** Find the real \( a \) knowing that the equation \( x^3 - 3x^2 + ax - 1 = 0 \) has three positive solutions. Solve the equation.
Hence they are real. The reals  

\[ \text{Let } x_1, x_2, x_3 \text{ be the three positive roots of the equation. Then } x_1 + x_2 + x_3 = 3 = 3\sqrt[3]{x_1x_2x_3}. \text{ But this equality holds only for } x_1 = x_2 = x_3 = 1. \text{ Hence } a = 3. \]

**Problem 10.35.** Show that the polynomial equation with real coefficients \( a_nx^n + a_{n-1}x^{n-1} + \ldots + a_3x^3 + x^2 + x + 1 = 0 \) has fewer than \( n \) real roots.

**Solution:** Let \( x_1, x_2, \ldots, x_n \) be the roots of the equation, and suppose that all of them are real. The reals \( y_1 = \frac{1}{x_1}, y_2 = \frac{1}{x_2}, \ldots, y_n = \frac{1}{x_n} \) are the solutions of the polynomial equation

\[ y^n + y^{n-1} + y^{n-2} + a_3y^{n-3} + \ldots + a_1y + a_0 = 0. \]

Then \( y_1^2 + y_2^2 + \ldots + y_n^2 = (y_1 + y_2 + \ldots + y_n)^2 - 2(y_1y_2 + y_1y_3 + \ldots + y_{n-1}y_n) = (-1)^2 - 2 \cdot 1 = -1, \) contradiction.

**Problem 10.36.** If \( x_1, x_2, x_3 \) are the complex roots of the polynomial equation

\[ x^3 - x^2 + x + 5 = 0 \]

evaluate the expression

\[ \frac{x_1}{1 + x_1} + \frac{x_2}{1 + x_2} + \frac{x_3}{1 + x_3}. \]

**Solution:** Denote \( \frac{x}{1 + x} = y, \) so \( x = \frac{y}{1 - y}. \) Then substituting in the equation we have

\[ \left( \frac{y}{1 - y} \right)^3 - \left( \frac{y}{1 - y} \right)^2 + \frac{y}{1 - y} + 5 = 0, \]

equation which can be written

\[ -2y^3 + 12y^2 - 14y + 5 = 0. \]

Then the expression to evaluate is

\[ y_1 + y_2 + y_3 = -\frac{12}{-2} = 6. \]

**Problem 10.37.** Consider all lines which meet the graph of \( y = 2x^4 + 7x^3 + 3x - 5 \) in four distinct points, say \( (x_i, y_i), i = 1, 2, 3, 4. \) Show that \( (x_1 + x_2 + x_3 + x_4)/4 \) is independent of the line, and compute its value.

**Solution:** Let \( y = ax + b \) such a line, so \( x_1, x_2, x_3, x_4 \) are solutions of the equation

\[ 2x^4 + 7x^3 + 3x - 5 = ax + b. \]

Therefore \( x_1 + x_2 + x_3 + x_4 = -\frac{7}{2}. \)

**Problem 10.38.** Let \( P(x) = x^3 + ax^2 + bx + c. \) Determine necessary and sufficient conditions such the roots of \( P(x) \) are

\[ a) \text{ in arithmetic progression} \]
\[ b) \text{ in geometric progression} \]

**Solution:** a) A necessary and sufficient condition is \( x_1 + x_3 = 2x_2, \) or \( x_2 = \frac{x_1 + x_2 + x_3}{3} = -a/3. \) The polynomial has the root \(-a/3\) if and only if \( P(-a/3) = 0, \) or \( c = \frac{ab}{3} - \frac{2a^3}{27}. \)
b) The condition is \( x_1 x_3 = x_2^2 \), which is equivalent with \( x_2 = \sqrt{c} \). The polynomial \( P \) has the root \(-\sqrt{c}\) if and only if \( c = \left( \frac{b}{a} \right)^3 \).

**Problem 10.39.** Suppose that \( a + \frac{1}{a} \in \mathbb{Q} \). Prove that \( a^n + \frac{1}{a^n} \in \mathbb{Q} \) for all integer \( n > 0 \).

### 3. Polynomials with rational coefficients

**Gauss’ Lemma**

Let \( F \) be a polynomial with integer coefficients. If \( F \) can be factored into a product of polynomials with rational coefficients, then \( F \) can be factored into a product of polynomials with integer coefficients.

**Problem 10.40.** If \( P \in \mathbb{Z}[x] \) and \( a \in \mathbb{Z} \), then \( P(a) \) divides \( P(a + P(a)) \).

**Problem 10.41.** Let \( P \in \mathbb{Z}[x] \). Prove there is \( n \in \mathbb{Z} \) such that \( P(n) \) is not a prime number.

**Problem 10.42.** Let \( P \in \mathbb{Q}[X] \) be a non-zero polynomial, and the sequence \( x_n = P(n) \). Prove that if the sequence \( x_n \) contains an integer then it has infinitely many integer terms.

**Solution:** Let \( P(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_1 x + a_0 \), and suppose \( P(m) \) is an integer. There is an integer \( p \) such that \( pa_i \) is an integer for any \( i \). Then \( P(m + np) = P(m) + \sum_{i=1}^{k} a_i [(m + np)^i - m^i] \) is an integer since \((m + np)^i - m^i\) is divisible by \( p \).

**Problem 10.43.** Suppose the complex number \( \alpha \neq 0 \) is a root of a polynomial of degree \( n \) with rational coefficients. Prove that \( 1/\alpha \) is also a root of a polynomial of degree \( n \) with rational coefficients.

**Solution:** If \( \alpha \) is the root of the polynomial \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \), then \( 1/\alpha \) is the root of \( Q(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \).

**Problem 10.44.** Let \( P(x) \) be a polynomial of degree \( n > 1 \) with integer coefficients. Let \( u(P) \) be the number of integers \( k \) such that \( P^2(k) = 1 \). Show that \( u(P) \leq 2 + n \).

**Solution:** For \( n = 2 \), \( u(P) \) is smaller than the degree of the polynomial \( P^2(x) \), so \( u(P) \leq 2 + n \). For \( n = 3 \), suppose that \( u(P) > 2 + n \). Then the polynomial equation \( P^2(x) = 1 \) has 6 roots, so each of the equations \( P(x) = 1 \) and \( P(x) = -1 \) has 3 solutions. If \( x_1 < x_2 < x_3 \) are the roots of the equation \( P(x) = 1 \), then \( P(x) = (x - x_1)(x - x_2)(x - x_3) + 1 \). Let \( y_1 < y_2 < y_3 \) be the roots of the equation \( P(x) = -1 \). We have \((y_1 - x_1)(y_1 - x_2)(y_1 - x_3) = -2 \), and the only factorisation of \(-2 \) in 3 distinct factors is \(-2 = (-1) \cdot 1 \cdot 2 \), so necessarily \( y_1 - x_1 = -1 \) and consequently \( y_1 = y_2 \), contradiction.

Consider now the case \( n > 3 \), and suppose \( u(P) \geq 3 + n \geq 7 \). We remark that the equations \( P(x) = 1 \) and \( P(x) = -1 \) have each at least 3 roots. Without
any restriction of the generality one can suppose that the number of roots of the equation \( P(x) = 1 \) is greater than the number of roots of the equation \( P(x) = -1 \). Then there are \( k_1, k_2, k_3, k_4 \) distinct integers such that \( P(k_i) = 1 \), for \( i = 1, 2, 3, 4 \) and also there is \( k \) such that \( P(k) = -1 \). There is a polynomial \( Q(x) \) with integer coefficients such that \( P(x) = (x - k_1)(x - k_2)(x - k_3)(x - k_4)Q(x) + 1 \), so \((k - k_1)(k - k_2)(k - k_3)(k - k_4)Q(k) = -2\). Contradiction, since \(-2\) cannot be written as the product of more than 3 distinct factors.

**Problem 10.45.** Give an example of a polynomial with rational coefficients, not all of them integers, which takes integer values for any integer.

**Problem 10.46.** For each \( i \geq 0 \) define a polynomial \( C_k(x) = \frac{1}{k!} x(x-1)(x-2)\ldots(x-(k-1)) \). Let \( f(x) \) be a polynomial with rational coefficients such that \( f(k) \) is an integer for each integer \( k \). Prove that there exist integers \( a_0, a_1, \ldots, a_n \) such that \( f(x) = \sum_{k=0}^{n} a_k C_k(x) \).

**Solution:** Let \( n = \deg P \). The polynomials \( C_0, C_1, \ldots, C_n \) constitute a basis of the linear space of polynomials of order \( \leq n \), so there are coefficients \( a_0, a_1, \ldots, a_n \) such that

\[
(3.1) \quad f(x) = \sum_{k=1}^{n} a_k C_k(x).
\]

It remains to prove that the numbers \( a_1, \ldots, a_n \) are integers. Taking \( x = 0 \) in the relation 3.1, we obtain \( f(0) = a_0 \) which shows that \( a_0 \) is an integer. We proceed by induction. We suppose that \( a_0, a_1, \ldots, a_{i-1} \) are all integers. Taking \( x = i \) in the equation 3.1 gives \( f(i) = \sum_{k=0}^{i} a_k C_k^i \). Using \( C_i^i = 1 \) we get \( a_i = f(i) - \sum_{k=0}^{i-1} a_k C_k^i \), so \( a_i \) is an integer.

**Problem 10.47.** Let \( P \in \mathbb{C}[x] \) be a polynomial of degree \( n \) such that \( P(0), P(1), P(2), \ldots, P(n) \) are integers. Prove that \( P(k) \) is an integer for any integer \( k \).

### 4. Polynomials with rational or integer roots

**Theorem** Let \( p, q \) be relatively prime integers, \( q \neq 0 \). If

\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0
\]

is a polynomial with integer coefficients such that \( \frac{p}{q} \) is a zero of \( P \), then \( p \) divides \( a_0 \) and \( q \) divides \( a_n \).

**Problem 10.48.** Let \( P(x) \) be a polynomial with integer coefficients such that \( P(3) = P(4) = -7 \). Prove that \( P(x) \) has no integer roots.

**Problem 10.49.** Show that if \( P(x) \) is a polynomial with integer coefficients, and there exists a positive integer \( k \) such that none of the integers \( P(1), P(2), \ldots, P(k) \) is divisible by \( k \) then \( P(x) \) has no integer root.
Problem 10.50. Given that $a, b, c$ are odd integers, prove that $ax^2 + bx + c = 0$ has no rational solution $x$.

Solution: Suppose there is a rational solution $\frac{p}{q}$ with $p, q$ integers with no common divisor greater than 1. Then $ap^2 + bpq + cq^2 = 0$. The numbers $p$ and $q$ cannot be both even, so we have the following possibilities: $p, q$ are both odd, $p$ is odd and $q$ is even, $p$ is even and $q$ is odd. In any of these cases the number $ap^2 + bpq + cq^2$ is odd, so cannot be 0.

Problem 10.51. Let $P(x)$ be a polynomial with integer coefficients such that $P(3) = P(4) = -7$. Prove that $P(x)$ has no integer roots.

Solution: Suppose $n$ is an integer root of $P(x)$. Then there is a polynomial $Q(x)$ with integer coefficients such that $P(x) = (x - n)Q(x)$. The numbers $3 - n$ and $4 - n$ must be divisors of $-7$ which is a contradiction, since one of $3 - n$ and $4 - n$ is even.

Problem 10.52. Show that if $P(x)$ is a polynomial with integer coefficients, and there exists a positive integer $k$ such that none of the integers $P(1), P(2), \ldots, P(k)$ is divisible by $k$ then $P(x)$ has no integer root.

Solution: Suppose $n$ is an integer root of $P(x)$. Then there is a polynomial $Q(x)$ with integer coefficients such that $P(x) = (x - n)Q(x)$. None of the integers $n - 1, n - 2, \ldots, n - k$ is divisible by $k$, since if one of them, say $n - p$, is divisible by $k$, then $P(p) = (p - n)Q(p)$ must also be divisible by $k$. But this is a contradiction, since between every $k$ consecutive integers there is one divisible by $k$.

Problem 10.53. Let $a, b, c$ be distinct integers. Prove there is no polynomial $P$ with integer coefficients such that $P(a) = b$, $P(b) = c$ and $P(c) = a$.

Solution: Suppose there is such a polynomial $P$. Since $m - n$ is a divisor of $P(m) - P(n)$ we have $a - b$ divisor of $P(a) - P(b) = b - c$ and similarly $a - c$ is divisor of $b - a$ and $b - c$ is a divisor of $c - a$. We obtain $a - b, b - c, c - a | a - b$, so $|a - b| = |b - c| = |c - a|$. If $a - b = c - b$ then $a = c$, contradiction. If $a - b = b - c$, then $a = \frac{b + c}{2}$ and consequently $\frac{|b - c|}{2} = |a - b| = |b - c|$, or $b = c$. Contradiction.

Problem 10.54. Let $P$ be a quadratic polynomial with integer coefficients. Suppose there are two integers $u$ and $v$ such that $P(u) = v$ and $P(v) = u$. Prove that the solutions of the equation $P(x) = x$ are not integers.

Solution: Let $P(x) = ax^2 + bx + c$. Subtracting the equations $P(u) = v$ and $P(v) = u$ and dividing by $u - v \neq 0$ we obtain $a(u + v) + b + 1 = 0$. Substituting $v$ from this relation into $P(u) = v$, and similarly substituting $u$ into $P(v) = u$ we see that $u$ and $v$ are the roots of the equation

$$ax^2 + (b + 1)x + c + \frac{b + 1}{a} = 0$$

Then the discriminant of this second order equation, $\Delta_1 = (b - 1)^2 - 4ac - 4$ is a perfect square, say $p^2$.

Suppose the equation $ax^2 + bx + c = x$ has rational roots. Then its discriminant $\Delta_2 = (b - 1)^2 - 4ac$ is the square of an integer $q^2$. Then $p^2 = q^2 - 4$ and the only
solutions in integers of this equation are \((0, -2)\) and \((0, 2)\). But \(p = 0\) means \(\Delta_1 = 0\) and consequently \(u = v\). Contradiction.

**Problem 10.55.** Let \(a, b, c, d\) be distinct integers such that the equation
\[(x - a)(x - b)(x - c)(x - d) - 4 = 0\]
has the integral root \(r\). Show that \(4r = a + b + c + d\).

**Solution:** We can suppose without any cost that \(d < c < b < a\). Then \(r - a = -2\), \(r - b = -1\), \(r - c = 1\) and \(r - d = 2\).

**Problem 10.56.** Let \(f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0\) be a polynomial with integer coefficients. Assume that \(a_0, a_1\) and \(f(1)\) are all odd. Show that \(f(x)\) has no rational root.

**Solution:** Suppose \(f\) has a root \(p/q\). Then \(p\) and \(q\) are both odd, since \(p\) divides \(a_0\) and \(q\) divides \(a_n\). There is a polynomial \(g\) with integer coefficients such that \(f(x) = (qx - p)g(x)\). Then \(f(1) = (q - p)g(1)\) is divisible by \(q - p\) which is even. Contradiction.

**Problem 10.57.** Let \(P \in Q[X]\) be a non-zero polynomial, and the sequence \(x_n = P(n)\). Prove that if the sequence \(x_n\) contains an integer then it has infinitely many integer terms.

**Problem 10.58.** Let \(a, b, c\) be distinct integers. Prove there is no polynomial \(P\) with integer coefficients such that \(P(a) = b\), \(P(b) = c\) and \(P(c) = a\).

**Problem 10.59.** Let \(f(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0\) be a polynomial with integer coefficients. Assume that \(a_0, a_1\) and \(f(1)\) are all odd. Show that \(f(x)\) has no rational root.

**Problem 10.60.** Let \(P\) be a polynomial with integer coefficients. If there are 5 distinct integers on which \(P\) takes the value 5, then there is no integer on which \(P\) takes the value 8.

5. Polynomials with real coefficients

**Problem 10.61.** Let \(P\) be a polynomial with real coefficients. If the equation \(P(x) = x\) doesn’t have any real roots, prove that the equation \(P(P(x)) = x\) also doesn’t have any real roots.

**Solution:** The continuous function \(f(x) = P(x) - x\) has no real zeros, hence it has constant sign. Without loss of generality we can assume \(f(x) < 0\) for all real values of \(x\). Then \(P(P(x)) < P(x) < x\) for any real \(x\).

**Problem 10.62.** Let \(p(x)\) be a polynomial that is nonnegative for all real \(x\). Prove that for some \(k\), there are polynomials \(f_1(x), \ldots, f_k(x)\) such that \(p(x) = \sum_{j=1}^{k} (f_j(x))^2\). Prove that one can take \(k = 2\).

**Problem 10.63.** Let \(f(x)\), \(g(x)\), \(h(x)\) be polynomials with real coefficients such that \(f^2(x) = g^2(x) + h^2(x)\). Assume that \(0 < \deg(g) < n\), where \(n\) is the degree of \(f(x)\). Show that \(f(x)\) has fewer than \(n\) real roots.
6. POLYNOMIALS WITH COMPLEX COEFFICIENTS

Complex-root theorem

If \( z \) is a complex root of a polynomial \( P \) with real coefficients, then \( \overline{z} \) is also a root of \( P \).

Problem 10.64. Consider the polynomial \( P(x) = x^5 + 5x^3 + 5x - 2m \), where \( m \) is a real number. Using the transformation \( x = u - 1/u \), find the real roots of \( P \).

Solution: We can use the transformation \( (0, \infty) \ni u \mapsto x = u - 1/u \in \mathbb{R} \), since its a bijection. We have \( x^3 = u^3 - 3u + \frac{3}{u} - \frac{1}{u^3} \), \( x^5 = u^5 - 5u^3 + 10u - \frac{10}{u} + \frac{5}{u^3} - \frac{1}{u^5} \).

The equation \( P(x) = 0 \) becomes \( u^5 - \frac{1}{u^5} - 2m = 0 \), with the only positive root \( u = \sqrt[5]{m} + \sqrt[5]{m^2 + 1} \). Therefore the only real root of \( P \) is \( x = \sqrt{m} + \sqrt{m^2 + 1} + \sqrt{m - \sqrt{m^2 + 1}} \).

Problem 10.65. Let \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 \) be a polynomial function with real coefficients, such that \( |a_n| \geq 1 \) and \( |P(i)| \leq 1 \). Prove that \( P \) has a root \( z_0 = a + bi \notin \mathbb{R} \) such that \( (a^2 + b^2 + 1)^2 \leq 4b^2 + 1 \).

Solution: Let \( x_1, x_2, \ldots, x_p \) be the real zeros and \( a_1 + b_1 i, a_1 - b_1 i, a_2 + b_2 i, a_2 - b_2 i, \ldots, a_q + b_q i, a_q - b_q i \) the complex zeros of \( P \) which has the decomposition \( P(x) = a_n(x - x_1)(x - x_2)(x - a_1 + b_1 i)(x - a_1 - b_1 i)(x - a_2 + b_2 i)(x - a_2 - b_2 i) \ldots (x - a_q + b_q i)(x - a_q - b_q i) \).

The condition \( |P(i)| \leq 1 \) writes \( a_n^2 (x_1^2 + 1)(x_2^2 + 1)(a_1^2 + (b_1)^2)(a_2^2 + (b_2)^2) \ldots (a_q^2 + (b_q)^2)(a_q^2 + (b_q - 1)^2) \leq 1 \). At least one of these factors must be smaller than 1, so there is \( k \) such that \( a_k^2 + (1 + b_k)^2 \leq 1 \) and this is the required condition.

6. Polynomials with complex coefficients

Complex-root theorem

If \( z \) is a complex root of a polynomial \( P \) with real coefficients, then \( \overline{z} \) is also a root of \( P \).

Problem 10.66. Let \( P \) be a polynomial with complex coefficients such that \( \text{Re} \, P(z) \leq \text{Im} \, P(z) \) for any complex \( z \). Then \( P \) is constant.

Solution: Suppose \( P \) is not constant. Then the equation \( P(z) = 1 \) has at least a complex solution \( z_0 \), and \( \text{Re} \, P(z_0) = 1 > \text{Im} \, P(z_0) = 0 \). Contradiction.

Problem 10.67. Let \( P(z) = z^3 + az + b \) be a polynomial with complex coefficients. If the roots \( z_1, z_2, z_3 \) have the same absolute value, then \( a = 0 \).

Solution: Let \( r = |z_1| = |z_2| = |z_3| \). Since \( z_1 + z_2 + z_3 = 0 \), the roots \( z_1, z_2, z_3 \) are the vertices of an equilateral triangle (see the chapter on complex numbers). There is a real number \( a \) such that \( z_1 = r (\cos a + i \sin a), z_2 = r (\cos \frac{a + 2\pi}{3} + i \sin \frac{a + 2\pi}{3}), z_3 = r (\cos \frac{a + 4\pi}{3} + i \sin \frac{a + 4\pi}{3}) \). A computation shows that \( a = z_1 z_2 + z_2 z_3 + z_3 z_1 = 0 \).

Problem 10.68. Let \( P(z) = z^2 + az + b \) be a quadratic polynomial of the complex variable \( z \) with complex coefficients \( a, b \), such that \( |P(z)| = 1 \) if \( |z| = 1 \). Prove that \( a = b = 0 \).

Solution: The condition \( |P(z)| = 1 \) writes \( (z^2 + az + b)(z^2 + az + b) = 1 \) or equivalently \( |a|^2 + |b|^2 + (\overline{a}z + \overline{b}z) + (b\overline{a}z + \overline{b}z^2) + (ba\overline{z} + \overline{b}z^2) = 0 \). Adding the four equations obtained successively for \( z = 1, z = -1, z = i \) and \( z = -i \) we get \( 4|a|^2 + 4|b|^2 = 0 \).
Problem 10.69. Let \( z \) be a complex number such that
\[
11z^{10} + 10iz^9 + 10iz - 11 = 0,
\]
where \( i^2 = -1 \). Prove that \( |z| = 1 \).

Solution: We rearrange the equation as
\[
z^9 = \frac{11 - 10iz}{11z + 10i}, \quad \text{so } |z|^9 = \left| \frac{11 - 10iz}{11z + 10i} \right| = \frac{121 + 100|z|^2 + 220\text{Im} z}{121|z|^2 + 100 + 220\text{Im} z}.
\]
Then \( |z|^9 - 1 = \frac{21(1 - |z|^2)}{121|z|^2 + 100 + 220\text{Im} z} \), and we can arrange this equation as
\[
(|z|^2 - 1) \left( |z|^{16} + |z|^{14} + \ldots + |z|^2 + 1 + \frac{21}{121|z|^2 + 100 + 220\text{Im} z} \right) = 0.
\]
The second factor is strictly positive, thus \( |z| = 1 \).

Problem 10.70. If \( a \in [-1, 1] \), and \( z \) is a complex number such that \( z^{n+1} - az^n + az - 1 = 0 \), prove that \( |z| = 1 \).

Solution: The equation can be arranged as \( z^n = \frac{1 - az}{z - a} \). Then
\[
|z|^{2n} = \frac{|1 - az|^2}{|z - a|^2} = \frac{1 + |a|^2|z|^2 - 2a\text{Re} z}{|z|^2 + |a|^2 - 2a\text{Re} z} = 1 + \frac{(1 - |a|^2)(|z|^2 - 1)}{|z - a|^2},
\]
so \( |z - a|^2(|z^{2n} - 1) + (1 - |a|^2)(|z|^2 - 1) = 0 \). The only positive root of this equation in \( |z| \) is 1.

7. Irreducible polynomials

Eisenstein’s Criterion

If all the coefficients of a polynomial, except the first, are divisible by a prime \( p \), and the constant coefficient is not divisible by \( p^2 \), then the polynomial cannot be factored as a product of two non-constant polynomials with integer coefficients.

Problem 10.71. Let \( a, b, c \) be integer numbers such that \( ac + bc \) is odd. Prove that the polynomial \( P(x) = x^3 + ax^2 + bx + c \) is irreducible in \( \mathbb{Z}[X] \).

Solution: Suppose the polynomial is the product of two polynomials with integer coefficients. One of these polynomials is of first degree with leading coefficient \( \pm 1 \), hence \( P(x) \) has an integer root \( n \). From \( (a + b)c \) odd, we have \( c \) odd, and \( a \), and \( b \) with different parity. Then for any parity of \( n \) the number \( P(n) \) is odd, which contradicts the fact that \( n \) is a root of \( P \).

Problem 10.72. Prove that every non-constant polynomial with integer coefficients is the sum of two irreducible polynomials with integer coefficients.

Solution: Let \( P(x) = \sum_{k=0}^{n} a_k x^k \). Define \( Q(x) = \sum_{k=0}^{n} b_k x^k \), with \( b_0 = 28(a_0 - 5) + 2 \), \( b_k = 4a_k \), for \( k = 1, n - 1 \), and \( b_n = 4a_n - 1 \), and also define \( R(x) = \sum_{k=0}^{n} c_k x^k \), with
c_0 = -27(a_0 - 5) + 3, c_k = -3a_k, for k = 1, n - 1, and c_n = -3a_n + 1. Then, by Eisenstein’s criterion for p = 2, respectively p = 3, the polynomial Q, respectively R is irreducible, and \( P = Q + R \).

**Problem 10.73.** Factor \( P(x) = x^6 + x^4 + 3x^2 + 2x + 2 \) over the integers into a product of irreducible factors.

**Solution:** We remark \( x^6 + x^4 + 3x^2 + 2x + 2 = x^6 + x^4 + x^2 + 2(x^2 + x + 1) = x^2(x^2 - x + 1)(x^2 + x + 1) + 2(x^2 + x + 1) = (x^2 + x + 1)(x^4 - x^3 + x^2 + 2) \). We prove that this is the factorization of \( P(x) \) in irreducible factors. It suffices to prove that \( Q(x) = x^4 - x^3 + x^2 + 2 \) is irreducible. The polynomial doesn’t have any factor of degree 1 since \( Q(x) > 0 \) for all \( x \). We suppose that \( Q(x) \) can be factored in two second degree polynomials. We distinguish the possibilities:

1. \( Q(x) = (x^2 + ax + 1)(x^2 + bx + 2) \). Developing the right side and identifying the coefficients we get the system
   
   \[
   \begin{align*}
   a + b &= -1 \\
   ab + 3 &= 1 \\
   2a + b &= 0
   \end{align*}
   \]
   which doesn’t have any solutions.

2. \( Q(x) = (x^2 + ax - 1)(x^2 + bx - 2) \). Similarly we get the system
   
   \[
   \begin{align*}
   a + b &= -1 \\
   ab - 3 &= 1 \\
   -2a - b &= 0
   \end{align*}
   \]
   which doesn’t have any solutions.

**Problem 10.74.** Is the polynomial \( x^5 - x^2 + 1 \) irreducible over the rationals?

**Problem 10.75.** For each integer \( m \), consider the polynomial

\[
P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.
\]

For what values of \( m \) is \( P_m(x) \) the product of two non-constant polynomials with integer coefficients?

**Problem 10.76.** If \( p \) is a prime number, then the polynomial \( P(x) = x^{p-1} + x^{p-2} + ... + x + 1 \) is irreducible over the integers.

**Problem 10.77.** Prove that the polynomial \( x^n - 120 \) is irreducible over the integers.

**Problem 10.78.** Let \( n \) be a positive integer and \( a_1, a_2, \ldots, a_n \) distinct integers. Prove that the polynomial

\[
P(x) = (x - a_1)^2(x - a_2)^2 \ldots (x - a_n)^2 + 1
\]

is irreducible in \( \mathbb{Z}[X] \).

**Problem 10.79.** Let \( n \) be a positive integer. Prove that the polynomial \( x^n + 5x^{n-1} + 3 \) is irreducible in \( \mathbb{Z}[X] \).
8. Equations in polynomials

The Identity Theorem

If two polynomials in $x$ of degree $\leq n$ take equal values for more than $n$ distinct values of $x$, then they are equal.

**Problem 10.80.** Prove that the only periodic polynomials with complex coefficients are constant.

**Solution:** Assume there is a non-constant periodic polynomial $P$. There is $T > 0$ such that $P(x) = P(x + T)$ for any $x \in \mathbb{C}$. Denote $n$ the degree of $P$. Since $P$ is non-constant we have $n > 1$ and the polynomial $P$ has exactly $n$ zeros. Let $a$ be one of the zeros of $P$. But then $a_k = a + kT$ is also a zero of $P$ for any $k \in \mathbb{Z}$. Contradiction.

**Problem 10.81.** Let $P$ be a polynomial over the real numbers such that $P(n) \geq 0$ and $P(m) \leq 0$ for infinitely many positive integers $n, m$. Prove that $P = 0$.

**Problem 10.82.** Let $a, b$ be real numbers. Find the polynomials satisfying $P(x) = P([x]) + P(x - [x])$, for any $x \in (a, b)$, where $[x]$ denotes the integer part of $x$.

**Solution:** In the case $(a, b) \subset (0, 1)$, we have $[x] = 0$ for any $x \in (a, b)$, and the polynomials satisfying the requirement are those with $P(0) = 0$. In the opposite case, there is $c \in (a, b)$ with $n = [c] \neq 0$. There is also $\epsilon > 0$ such that $c + \epsilon < b$, and $[x] = n$ for any $x \in (c, c + \epsilon)$. For any such $x$ we have $P(x) = P(n) + P(x - n)$. But two polynomials equal for infinitely many values of $x$ are identical. Hence $P(x) = P(n) + P(x - n)$, for any real $x$. Taking the derivative $P'(x) = P'(x - n)$, but the only periodic polynomials are constant, so $P'(x) = \alpha$. For $x = 0$ in the equation, we obtain $P(0) = 0$, therefore $P(x) = \alpha x$.

**Problem 10.83.** Find the polynomials divisible by the sum of their derivatives.

**Solution:** If $P$ is a polynomial of degree $n$ divisible by the sum of the derivatives, then there are constants $a, b$ such that $P(x) = (P'(x) + P''(x) + \ldots + P^{(n)}(x))(ax + b)$. Taking the derivative we obtain $P'(x) = (P''(x) + \ldots + P^{(n)}(x))(ax + b) + aP'(x) + P''(x) + \ldots + P^{(n)}(x)) = \left(\frac{P(x)}{ax + b} - P'(x)\right)(ax + b) + a\frac{P(x)}{ax + b}$. We can arrange this equation as

\begin{equation}
(8.1) \quad P(x)(ax + a + b) = P'(x)(ax + b)(ax + b + 1)
\end{equation}

Let $a_n$ be the leading coefficient of $P(x)$. Identifying the leading coefficients in the relation 8.1 we obtain $a_n a = na_n a^2$, thus $a = \frac{1}{n}$. The differential equation 8.1 has the solution $P(x) = k(ax + b)(ax + b + 1)^{\frac{n-2}{2}} = k \left(\frac{x}{n} + b\right)^{\frac{n-2}{2}} \left(\frac{x}{n} + b + 1\right)^{n-1}$.

With $c = b + 1$ the general solution of the problem is necessarily of the form $P(x) = k \left(\frac{x}{n} + c\right)^n - k \left(\frac{x}{n} + c\right)^{n-1}$. To prove that this polynomial is divisible by the sum of its derivatives, denote $Q(x) = \left(\frac{x}{n} + c\right)^n$ and $R(x) = \left(\frac{x}{n} + c\right)^{n-1}$.
Using the remark $Q^{(n+1)}(x) = R^{(m)}(x)$, for any positive integer, we obtain $P'(x) + P''(x) + ... + P^{(n)}(x) = kQ'(x)$ which is a divisor of $P(x)$.

**Problem 10.84.** Prove there is no non-zero polynomial $P$ satisfying

$$P(x + 2) + P(x) = \sqrt{2}P(x + 1).$$

**Solution:** We have $P(x + 3) = \sqrt{2}P(x + 2) - P(x + 1) = P(x + 1) - \sqrt{2}P(x)$, and $P(x + 4) = P(x + 2) - \sqrt{2}P(x + 1) = -P(x)$. Then $P(x + 8) = P(x)$. But the only periodic polynomials are the constants, and the only constant satisfying the equation is 0.

**Problem 10.85.** Determine the polynomials $P$ with integer coefficients, such that $P(1) = 0$, and $0 < P(n) < n$, for any integer $n > 1$.

**Solution:** There is a polynomial $Q$ with integer coefficients, such that $P(x) = (x - 1)Q(x)$. Then $0 < (n - 1)Q(n) < n$, or $0 < Q(n) < 1 + \frac{1}{n - 1}$, for any integer $n > 1$. But $Q(n)$ is an integer, and $1 + \frac{1}{n - 1} \leq 2$, hence $Q(n) = 1$, for any integer $n > 1$. Then $Q$ is the constant polynomial 1, and $P(x) = x - 1$.

**Problem 10.86.** Determine the polynomials $P$ such that

$$P(x)P(1 - x) = P(1 - x^2).$$

**Solution:** The only constant polynomials satisfying the equation are 0 and 1. Suppose there is a non-constant solution $P(x) = a_n x^n + a_{n-1} x^{n-1} + ...$. Then $P(x)P(1 - x) = a_n^2 (-1)^n x^{2n} + a_n^2 (-1)^n x^{2n-1} + ...$. But in $P(1 - x^2)$ the coefficients of the odd terms are 0, so $a_n = 0$. Contradiction.

**Problem 10.87.** Determine the polynomials $P$ satisfying the equation

$$P(x)P(x - 1) = P(x^2).$$

**Solution:** If $z$ be a root of $P$, then $z^2$ is also a root. By recurrence, $z^{2^n}$ is a root, for any integer $n \geq 1$. Since $P$ has finitely many roots, there are $m > n$ such that $z^{2^m} = z^{2^n}$. Then, for $p = 2^m - 2^n$ we have $z^p = 1$, thus $|z| = 1$. Also $P((z + 1)^2) = P(z)P(z + 1) = 0$, hence $(z + 1)^2$ is a root of $P$, and $|z + 1|^2 = 1$. This equation can be arranged $1 = (z + 1)(\bar{z} + 1) = |z|^2 + z + \bar{z} + 1$, or $0 = z + \bar{z} + 1 = z + \frac{z^2}{z} + 1 = z + \frac{1}{z} + 1$. We proved that any root $z$ of $P$ satisfies $z^2 + z + 1 = 0$, so there is a positive integer $k$ such that $P(x) = (x^2 + x + 1)^k$.

**Problem 10.88.** Let $k > 1$ be an integer. Find the polynomials $P$ satisfying $P(0) = 0$ and $P(x^k + 1) = P(x)^k + 1$, for any real $x$.

**Solution:** Define the sequence $(x_n)_n$ by $x_0 = 0$, $x_{n+1} = x_n^k + 1$. We prove by induction that $P(x_n) = x_n$, for any positive integer $n$. Since the sequence $(x_n)_n$ is increasing, the polynomial $Q(x) = P(x) - x$ has infinitely many roots, therefore $Q$ is the polynom 0, and $P(x) = x$.

**Problem 10.89.** Let $a, b$ be real numbers. Determine the polynomials $P$ satisfying

$$x(P(x) - b) = (x - a)P(x + a).$$
10. POLYNOMIALS

**Problem 10.90.** Determine all polynomials $P(x)$ such that $P(0) = 0$ and $P(x^2 + 1) = P^2(x) + 1$.

**Solution:** There is a polynomial $Q(x)$ such that $P(x) = xQ(x)$. We replace in the second condition and we get $(x^2 + 1)Q(x^2 + 1) = x^2Q^2(x) + 1$. Taking $x = 0$ and $x = 1$ in this relation we see that $Q(1) = Q(2) = 1$. Then there is a polynomial $R(x)$ such that $Q(x) = (x - 1)(x - 2)R(x) + 1$ and replacing in the equation gives $(x - 1)(x^2 + 1)R(x^2 + 1) = (x - 2)R(x) [(x - 1)(x - 2)R(x) + 2]$. We consider the sequence $(x_n)$ defined by $x_0 = 2$ and $x_{n+1} = x_n^2 + 1$. One can easily prove by induction that $R(x_n) = 0$ for any $n > 0$, so $R(x) = 0$ as a polynomial with infinitely many roots. Replacing this back we obtain $Q(x) = 1$ and $P(x) = x$.

**Problem 10.91.** Let $k$ be a positive integer. Find all polynomials $P(x)$ with real coefficients such that $P(P(x)) = P(x)^k$.

**Solution:** The only solutions are $P_1 \equiv 0$ and $P_2(x) = x^k$. Of course $P_1$ is a solution. Suppose now $P$ is a solution not identically zero. Then necessarily deg $P = k$. Let $P(x) = a_kx^k + a_{k-1}x^{k-1} + \ldots + a_1x + a_0$. Then the equation becomes $a_kP(x)^k + a_{k-1}P(x)^{k-1} + \ldots + a_1P(x) + a_0 = P(x)^k$. A first consequence is that $a_{k+1} = a_k$, so $a_k = 1$. Therefore $a_{k-1}P(x)^{k-1} + \ldots + a_1P(x) + a_0 = 0$, for any $x$, which forces $a_{k-1} = a_{k-2} = \ldots = a_1 = a_0 = 0$, since the polynomials $P(x)^{k-1}, P(x)^{k-2} \ldots P(x), 1$ have different degrees so they are independent.

**Problem 10.92.** Let $n > 1$ be an integer. Find the polynomials $P(x)$ satisfying $P(x^n) = P^n(x)$ for every real $x$.

**Solution:** If $P(x) = a_kx^k + \ldots + a_1x + a_0$, identifying the leading coefficients in the equation we get $a_k = a_k^n$. If $a_k = 0$, then $P(x) = 0$. If $a_k = 1$, there is a polynomial $Q(x)$ of degree $p \leq k - 1$ such that $P(x) = x^k + Q(x)$. The equation becomes $x^{nk} + Q(x^n) = (x^k + Q(x))^n$, or $Q(x^n) = n\alpha_k^{k-1}Q(x) + \ldots + Q^n(x)$. Identifying the degrees $np = k(n - 1) + p$, and this can happen only for $p = -\infty$. Therefore $Q(x) = 0$ and $P(x) = x^k$.

**Problem 10.93.** Find the polynomials $P(x)$ with real coefficients having the property $P(Q(x)) = Q(P(x))$ for every polynomial $Q(x)$ with real coefficients.

**Solution:** Taking $Q(x) = \alpha x$, we obtain $P(\alpha x) = \alpha P(x)$ for every $a, x$, and for $Q(x) = x + \beta$, we get $P(x + \beta) = P(x) + \beta$. The only polynomial with these two properties is $P(x) = x$.

**Problem 10.94.** Let $n > 2$ be an integer. Find all the polynomials $P$ with complex coefficients satisfying $P(0) = 1$, and $P(x_1) + P(x_2) + \ldots + P(x_n) = 0$, whenever $x_1 + x_2 + \ldots + x_n = 1$.
Solution: For \( x_1 = x_2 = \ldots = x_n = \frac{1}{n} \), we have \( P \left( \frac{1}{n} \right) = 0 \). There is a polynomial \( Q(x) \) such that \( P(x) = \left( x \frac{1}{n} \right) Q(x) \), and this polynomial satisfies

\[
\sum_{k=1}^{n} (nx_k - 1)Q(x_k) = 0, \text{ if } \sum_{k=1}^{n} x_k = 1
\]

We remark \( Q(0) = -n \). Taking \( x_1 = 1, x_2 = \ldots = x_n = 0 \) in 8.2, we obtain \( Q(1) = Q(0) = -n \). Again in 8.2, with \( x_1 = -1, x_2 = x_3 = 1, x_4 = \ldots = x_n = 0 \), we have \( -(n+1)Q(-1) + 2(n-1)Q(1) - (n-3)Q(0) = 0 \), and from this \( Q(-1) = -n \).

We prove by induction that \( Q(k) = -n \), for any \( k \) integer. More precisely, we will prove that \( Q(k) = Q(-k) = -n \) implies \( Q(k+1) = Q(-k-1) = -n \). And for this we take first \( x_1 = k+1, x_2 = -k, x_3 = \ldots = x_n = 0 \) in 8.2 and then \( x_1 = -k-1, x_2 = k+1, x_3 = 1, x_4 = \ldots = x_n = 0 \). The solution is \( P(x) = -nx+1 \).

Problem 10.95. a) Find the polynomials \( P(x) \) satisfying

\[
(x+2)P(x) = xP(x+1).
\]

b) Find the polynomials \( Q(x) \) satisfying \((x-2)Q(x) = xQ(x+1)\).

Solution: a) Since \( P(0) = P(-1) = 0 \), there is a polynomial \( P_1(x) \) such that \( P(x) = x(x+1)P_1(x) \). Then \( P_1(x) = P_1(x+1) \), so \( P_1 \) is constant.

b) Taking \( x = 0 \) we get \( Q(0) = 0 \), so there is a polynomial \( Q_1(x) \) such that \( Q(x) = xQ_1(x) \). The equation becomes \((x-2)Q_1(x) = (x+1)Q_1(x+1) \). By induction we prove that for any \( n \) there is a polynomial \( Q_n(x) \) such that \( Q(x) = x(x+1)\ldots(x+n-1)Q_n(x) \). Hence \( Q \) is the constant polynomial 0.

Problem 10.96. Determine the polynomials \( f \in \mathbb{R}[x] \) such that \( f(a+b) = f(a) + f(b) \), for any reals \( a \) and \( b \).

Problem 10.97. Determine all the polynomials \( P \) of degree \( n \) satisfying

\[
P(x) = P'(x)P''(x)\ldots P^{(n-1)}(x)P^{(n)}(x).
\]

9. Rational functions

Problem 10.98. If \( f(x) = \frac{P(x)}{Q(x)} \) is a rational function, with \( P, Q \) polynomials such that \( Q \) has no real zeros and \( \deg P \leq \deg Q \), prove that there are finitely many real numbers \( x \) for which \( f(x) \) is an integer.

Solution: The function \( f \) is bounded so there are only finitely many integers values that \( f \) can take and each of these integer values can be attained only for finitely many values of \( x \).

Problem 10.99. Prove there is no rational function \( f \) such that \( f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \), for \( n \geq 1 \).
Solution: Suppose there are polynomials $P$ and $Q$ such that \( \frac{P(n)}{Q(n)} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \). Since \( \lim_{n \to \infty} \frac{P(n)}{Q(n)} = \infty \), \( \deg P > \deg Q \). On the other side \( \lim_{n \to \infty} \frac{n P(n)}{nQ(n)} = 0 \) (for example use Cesaro-Stolz), so \( \deg P < 1 + \deg Q \). Contradiction.

**Problem 10.100.** (i) A rational function \( R \) is even if and only if there is a rational function \( R_1 \) such that \( R(x) = R_1(x^2) \).

(ii) A rational function \( R \) is odd if and only if there is a rational function \( R_1 \) such that \( R(x) = xR_1(x^2) \).

Solution: (i) If \( P \) is a polynomial which is an even function then there is a polynomial \( P_1 \) such that \( P(x) = P_1(x^2) \). Let \( R(x) = \frac{P(x)}{Q(x)} \) with \( P, Q \) polynomials.

But \( P(x)Q(-x)P(-x)Q(x) \) and \( Q(x)Q(-x) \) are even polynomials, so there are polynomials \( P_1, Q_1 \) such that \( R(x) = \frac{P_1(x^2)}{Q_1(x^2)} \).

(ii) The rational function \( \frac{R(x)}{x} \) is even, so using (i) there is a rational function \( R_1 \) such that \( \frac{R(x)}{x} = R_1(x^2) \).

**Problem 10.101.** Let \( f(x) = \frac{ax^4 + bx^3 + cx^2 + dx + e}{a_1x^4 + c_1x^2 + e_1} \). Prove that \( f \) is constant if and only if \( b = d = 0 \) and \( \frac{a}{a_1} = \frac{c}{c_1} = \frac{e}{e_1} \).

Solution: Suppose the function is constant. Then \( \frac{e}{e_1} = f(0) = \lim_{x \to \infty} f(x) = \frac{a}{a_1} \).

From \( f(x) = f(-x) \) for every \( x \) we obtain that \( bx^3 + dx = 0 \) for all \( x \), which is possible only if \( b = d = 0 \). Let \( k \) be such that \( a = ka_1 \) and \( e = ke_1 \). The function has then the form \( f(x) = k + (c - kc_1) \frac{x^2}{a_1x^4 + c_1x^2 + e_1} \) and such a function is constant only if \( c = kc_1 \).

The other implication is straightforward.

**Problem 10.102.** For what positive integer \( n \) is \( x^n + \frac{1}{x^n} \) expressible as a polynomial with real coefficients in \( x - \frac{1}{x} \)?

Solution: As a consequence of the identity

\[
y^m + \frac{1}{y^m} = \left(y^{m-1} + \frac{1}{y^{m-1}}\right) \left(y + \frac{1}{y}\right) - \left(y^{m-2} + \frac{1}{y^{m-2}}\right)
\]
\[ y^m + \frac{1}{y^m} \] can be expressed as a polynomial in \( y + \frac{1}{y} \) for any integer \( m \). If \( n \) is even, say \( n = 2k \), then \( (x^2)^k + \frac{1}{(x^2)^k} \) can be expressed as a polynomial in \( x^2 + \frac{1}{x^2} = \left( x + \frac{1}{x} \right)^2 - 2 \), and consequently as a polynomial in \( x - \frac{1}{x} \).

If \( n \) is odd, suppose there is a polynomial \( P \) such that \( x^n + \frac{1}{x^n} = P \left( x - \frac{1}{x} \right) \). The degree of \( P \) must be \( n \) and the leading coefficient 1. But then \( \lim_{x \to 0^+} \left( x^n + \frac{1}{x^n} \right) = \infty \) and \( \lim_{x \to 0^+} P \left( x - \frac{1}{x} \right) = \lim_{y \to -\infty} P(y) = -\infty \). The contradiction shows that the answer of the problem is \( n \) even.

**Problem 10.103.** Find the minimum value of

\[
\frac{(x + 1/x)^6 - (x^6 + 1/x^6) - 2}{(x + 1/x)^3 + (x^3 + 1/x^3)}
\]

for \( x > 0 \). [P1998]

**Solution :** Let \( y = x + \frac{1}{x} \). Then

\[
x^6 + \frac{1}{x^6} = (x^2 + \frac{1}{x^2}) \left( x^4 - \frac{1}{x^4} \right) = (y^2 - 2)((y^2 - 2)^2 - 3)
\]

eetc. and our expression becomes \( \frac{y^6 - (y^2 - 2)^3 + 3(y^2 - 2) - 2}{y^3 + y^3 - 3y} = 3y \). The minimum is then 6 and is obtained for \( x = 2 \).

10. Polynomials of several variables

**Problem 10.104.** Let \( P \) be a polynomial, with real coefficients, in three variables and \( F \) be a function of two variables such that

\[ P(ux, uy, uz) = u^2F(y - x, z - x) \quad \text{for all real } x, y, z, u, \]

and such that \( P(1, 0, 0) = 4, P(0, 1, 0) = 5, \) and \( P(0, 0, 1) = 6 \). Also let \( A, B, C \) be complex numbers with \( P(A, B, C) = 0 \) and \( |B - A| = 10 \). Find \( |C - A| \). [P1987]

**Solution :** We remark that \( F(y, z) = P(0, y, z) \), so \( F \) is a polynomial in two variables. Taking \( x = 0 \) in the definition equation we obtain \( F(uy, uz) = u^2F(y, z) \). But then \( F \) must contain only second order monomials, so \( F(y, z) = ay^2 + byz + cz^2 \). Consequently, \( P(x, y, z) = a(y - x)^2 + b(y - x)(z - x) + c(z - x)^2 \) and we determine the coefficients using the conditions \( P(1, 0, 0) = 4, P(0, 1, 0) = 5, P(0, 0, 1) = 6 \). The polynomial has then the form \( P(x, y, z) = 5(y - x)^2 - 7(y - x)(z - x) + 6(z - x)^2 \), and the condition \( P(A, B, C) = 0 \) writes \( 6z^2 - 7z + 5 = 0 \), where \( z = \frac{C - A}{B - A} \). We solve the equation, \( z = \frac{7 \pm i\sqrt{11}}{12} \), \( |z| = \frac{\sqrt{120}}{12} \). Hence \( |C - A| = 10|z| \).
10. Polynomials

10.105. Let \( f(x, y, z) = x^2 + y^2 + z^2 + xyz \). Let \( p(x, y, z), q(x, y, z), r(x, y, z) \) be polynomials with real coefficients satisfying
\[
f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).
\]
Prove or disprove the assertion that the sequence \( p, q, r \) consists of some permutation of \( \pm x, \pm y, \pm z \), where the number of minus signs is 0 or 2. [P1986]

**Solution**: A counterexample is \( p = x, q = y, r = -z - xy \).

10.106. Let \( \mathbb{C}^3 \) denote the set of ordered triples of complex numbers. Define a map \( F : \mathbb{C}^3 \to \mathbb{C}^3 \) by \( F(u, v, w) = (u + v + w, uv + vw + wu,uvw) \). Prove that \( F \) is onto but not one-to-one.

**Solution**: Let \( (a, b, c) \in \mathbb{C}^3 \). The three zeros \( (x_1, x_2, x_3) \) of the polynomial \( P(x) = x^3 - ax^2 + bx - c \) are satisfying the condition \( F(x_1, x_2, x_3) = (a, b, c) \), so the function \( F \) is onto. Since \( f \) is symmetrical in the three variables it cannot be one-to-one.

10.107. Find a non-zero polynomial \( P(x, y) \) such that \( P([a], [2a]) = 0 \) for all real number \( a \).

**Solution**: From \( |a| \leq a < |a| + 1 \) we have \( 2|a| \leq 2a < 2|a| + 2 \), and as a consequence \( 2a \) can be either \( 2|a| \) or \( 2|a| + 1 \). Then we can take \( f(x, y) = (y - 2x)(y - 2x - 1) \).

10.108. Prove there are no polynomials of one variable with complex coefficients \( P, Q, R, S \) all of them of degree at least one such that
\[
P(x)Q(y) - R(x)S(y) = 1, \forall x, y \in \mathbb{R}
\]

**Solution**: Since \( \deg S \geq 1 \) there exist \( a \in \mathbb{C} \) such that \( S(a) = 0 \). We substitute \( y = a \) in the equation and we obtain \( P(x)Q(a) = 1 \), thus \( P(x) = \frac{1}{Q(a)} \). Contradiction with the fact that \( \deg P \geq 1 \).

11. Multiple roots

10.109. If \( n > 1 \), show that \( P(x) = (x + 1)^n - x^n - 1 \) has a multiple zero if and only \( n - 1 \) is divisible by 6.

**Solution**: The polynomial \( P(x) \) has a multiple zero if and only if there is a \( z \) such that \( P(z) = 0 \) and \( P'(z) = 0 \), or equivalently \( (z + 1)^{n-1} = z^{n-1} = 1 \).

Suppose there exist such a \( z \). From \( z^{n-1} = 1 \) it follows that there is \( k \in \{0, 1, 2, \ldots, n - 2\} \) such that \( z = \cos \frac{2k\pi}{n-1} + i \sin \frac{2k\pi}{n-1} \). Then \( (z + 1)^{n-1} = 2^{n-1} \cos^{n-1} \frac{k\pi}{n-1} \cos k\pi \). We distinguish the cases:

1. \( k = 2p \). In this case \( 2 \cos \frac{2p\pi}{n-1} = 1 \), so \( \frac{2p\pi}{n-1} = \frac{\pi}{3} \), i.e. \( n - 1 = 6p \).

2. \( k = 2p + 1 \). In this case \( 2 \cos \frac{2p\pi}{n-1} \) must be \(-1\) and therefore \( \frac{(2p + 1)\pi}{n-1} = \frac{2\pi}{3} \) or \( n - 1 = \frac{6p + 3}{2} \) which is a contradiction.
Conversely, if \( n - 1 = 6p \), then \( z = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \) which has the property \((z + 1)^6 = z^6 = 1\), satisfies also \((z + 1)^{n - 1} = z^{n-1} = 1\)

**Problem 10.110.** The number \( a \) is a zero of multiplicity \( q \) of the polynomial \( F \) if and only if \( F(a) = F'(a) = \ldots = F^{(q-1)}(a) = 0 \) and \( F^{(q)} \neq 0 \).

**Problem 10.111.** If \( n \geq 2 \) is an integer, prove that the polynomial \( P(x) = (x + 1)^n - x^n - 1 \) has a multiple root if and only if \( n - 1 \) is a multiple of 6.

**Problem 10.112.** Prove that the polynomial \( P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \) doesn't have multiple zeros. Generalization.

**Problem 10.113.** The polynomial \( P(x) = x^5 + ax^3 + b \) has a double zero. Find the relation between \( a \) and \( b \).

### 12. Varia

**Problem 10.114.** Let \( a_0, a_1, \ldots, a_{16} \) be real numbers such that \((1+n-2n^2)^8 = \sum_{k=0}^{16} a_k n^k\), for any positive integer \( n \). Determine \( a_0 + a_1 + \ldots + a_{16} \) and \( a_2 \).

**Solution:** The polynomial \( P(x) = (1 + x - 2x^2)^8 \) has the coefficients \( a_0, a_1, \ldots, a_{16} \). Then \( a_0 + a_2 + \ldots + a_{16} = (P(1) + P(-1))/2 = 128 \), and \( a_2 = P''(0)/2 = 12 \).

**Problem 10.115.** Let \( P_1, P_2, \ldots, P_n \) be polynomials of degree \( n - 2 \). If any \( n - 1 \) of these polynomials have a common root, prove that all \( n \) polynomials have a common root.

**Solution:** Suppose there is no common root of all \( n \) polynomials. Then the common roots of any two subsets of \( n - 1 \) polynomials are distinct. For \( 1 \leq i \leq n - 1 \), denote by \( x_i \) the common root of polynomials in the set \( \{P_1, P_2, \ldots, P_n\} \setminus \{P_i\} \). Then \( x_1, x_2, \ldots, x_{n-1} \) are distinct roots of \( P_n \). Contradiction, since \( P_n \) has degree \( n - 2 \).

**Problem 10.116.** Prove that for any integer \( m \geq 0 \) the sum \( S_m(n) = \sum_{k=1}^{n} k^{2m+1} \) is a polynomial in \( n(n+1) \).

**Problem 10.117.** Find a necessary and sufficient condition such that the polynomials \( P(x) = x^2 + px + q \) and \( Q(x) = x^2 + px + q \) have exactly one common root. Find a quadratic polynomial whose roots coincide with the remaining roots of \( P \) and \( Q \).

**Solution:** The polynomials \( P \) and \( Q \) have exactly one common root if and only if \( a \neq p \) and

\[
(q - b)^2 + a(q - b)(a - p) + b(a - p)^2 = 0
\]

Suppose that \( P \) and \( Q \) have one common root. Let \( \{x_0, x_1\} \) be the roots of \( P \) and \( \{x_0, x_2\} \) be the roots of \( Q \), with \( x_1 \neq x_2 \). Then \( a = -x_0 - x_1 \neq x_0 - x_2 = p \). Subtracting the equations \( x_0^2 + ax_0 + b = 0 \) and \( x_0^2 + px_0 + q = 0 \) we obtain \( x_0 = \frac{q - b}{a - p} \) and replacing back in \( x_0^2 + ax_0 + b = 0 \) we get the condition 12.1.
The relations \( x_0 x_1 = b \) and \( x_0 x_2 = q \) give \( x_1 = \frac{b(a-p)}{q-b} \) and \( x_2 = \frac{q(a-p)}{q-b} \).

The quadratic polynomial with roots \( x_1 \) and \( x_2 \) is \( R(x) = x^2 - \alpha x + \beta \), where
\[
\alpha = x_1 + x_2 = \frac{(b+q)(a-p)}{q-b} \quad \text{and} \quad \beta = x_1 x_2 = \frac{bq(a-p)}{q-b}.
\]

Suppose \( a \neq p \) and the condition 12.1 satisfied. Then \( x_0 = \frac{q-b}{a-p} \) is a common root of \( p \) and \( Q \), since \( P(x_0) = \frac{(q-b)^2 + a(q-b)(a-p) + b(a-p)^2}{(a-p)^2} = 0 \) and \( Q(x_0) = \frac{(q-b)^2 + a(q-b)(a-p) + b(a-p)^2}{(a-p)^2} = 0 \). This is the only common root, otherwise \( a = p \).

**Problem 10.118.** Evaluate \( Q(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k}(x-k)^n \).

**Solution:** Let \( \mathcal{P} = \mathbb{R}[x] \) be the set of polynomials with real coefficients and define \( A : \mathcal{P} \to \mathcal{P} \), \( (Ap)(x) = p(x) - p(x-1) \). We prove by induction that
\[
(12.2) \quad (A^n p)(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} p(x-k)
\]
For \( n = 1 \), 12.2 is just the definition of \( A \). Suppose 12.2 satisfied for \( n \). Then
\[
(A^{n+1} p)(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (Ap)(x-k)
\]
\[
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (p(x-k) - p(x-k-1))
\]
\[
= \sum_{k=0}^{n} (-1)^k \binom{n}{k} p(x-k) + \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} p(x-k-1)
\]
\[
= p(x) + \sum_{k=1}^{n} \left( \binom{n}{k} + \binom{n}{k-1} \right) p(x-k) + (-1)^{n+1} p(x-n-1)
\]
\[
= \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} p(x-k).
\]
We notice that \( Q(x) = (Ap)(x) \), for \( p(x) = x^n \). But if \( p \) is a polynomial of degree \( n \) and leading coefficient \( a_n \), then a simple computation shows that \( Ap \) is a polynomial of degree \( n-1 \) with leading coefficient \( na_n \), and by recurrence \( A^k p \) is a polynomial of degree \( n-k \) with leading coefficient \( n(n-1)\ldots(n-k+1)a_n \). In particular, for \( p(x) = x^n \) and \( k = n \) we get \( (A^n p)(x) = n! \).

**Problem 10.119.** Consider the functions defined by \( T_n(x) = \cos(n \arccos x) \) for \( n \in \mathbb{N} \).
(a) Show that \( T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \), for \( n \geq 1 \).
(b) Show that \( T_n(x) \) is a polynomial function of degree \( n \) (called Chebyshev polynomial).
(c) What are the zeros of \( T_n \)?
(d) At what numbers does \( T_n \) have local maximum and minimum values?
(e) Evaluate the integral \( \int_{-1}^{1} T_n(x)\,dx \).

**Solution:**

(a) \( T_{n+1}(x) + T_{n-1}(x) = \cos((n + 1) \arccos x) + \cos((n - 1) \arccos x) = 2\cos(n \arccos x) \cos(\arccos x) = 2xT_n(x) \)

(b) Using part (a) and \( T_0(x) = 1, T_1(x) = x \) one proves easily by induction that \( T_n \) is a polynomial function of degree \( n \). The function \( T_n \) is defined on \([-1,1]\) with values in \([-1,1]\), but can be extended to \( \mathbb{R} \).

(c) The equation \( T_n(x) = 0 \) gives \( n \arccos x = k\pi + \frac{\pi}{2} \), with \( k \in \mathbb{Z} \). But \( \arccos x \in [0,\pi] \), so \( k \) can take only values from \( \{0,1,2,\ldots,n-1\} \). Therefore, the \( n \) zeros of \( T_n \) are \( \cos \left( \frac{2k+1}{2n} \pi \right), \quad k = 0,1,2,\ldots,n-1 \).

(d) Since \( T_n'(x) \) is a polynomial of order \( n - 1 \), the function \( T_n \) can have at most \( n - 1 \) extremum points. We prove that all these points lie in the interval \((-1,1)\). The extremum points on the interval \((-1,1)\) are among the critical points \( T_n'(x) = \sin(n \arccos x) \frac{n}{\sqrt{1-x^2}} = 0 \). The solutions of this equation are \( x_k = \cos \frac{k\pi}{n}, \quad k = 1,2,\ldots,n-1 \). But \( T_n(x_k) = (-1)^k \), and for \( x \in [-1,1] \) the polynomial \( T_n(x) \) takes values in \([-1,1]\), so all the \( x_k \)'s are extremum points. Since \( T_n \) can have at most \( n - 1 \) extremum points, these are all the extremum points.

(e) The substitution \( u = \arccos x \) gives \( \int_{-1}^{1} T_n(x)\,dx = \int_{0}^{\pi} \cos nu \sin u \,du = \frac{1 + (-1)^n}{1 - n^2} \). If \( n \) is odd, the integral is 0, and if \( n \) is even, the integral is \( \frac{2}{1 - n^2} \).

**Problem 10.120.** The Bernoulli polynomials \( B_n, \ n \in \mathbb{N}, \) are defined by \( B_0(x) = 1, \ B_1'(x) = B_{n-1}(x), \) and \( \int_{0}^{1} B_n(x)\,dx = 0 \) for \( n \in \mathbb{N} \). The Bernoulli numbers are \( b_n = n!B_n(0) \).

(a) Show that \( B_n(x) = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} b_k x^{n-k} \)

(b) \( b_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} b_k \)

(c) \( b_{2n+1} = 0 \) for \( n > 1 \).

(d) \( B_{n+1}(x+1) - B_{n+1}(x) = \frac{x^n}{n!} \).

(e) \( \sum_{m=0}^{k} m^n = n!(B_{n+1}(k+1) - B_{n+1}(0)) = n! \int_{0}^{k+1} B_n(x)\,dx \)

(f) Find a formula for \( 1^4 + 2^4 + 3^4 + \ldots + k^4 \) and \( 1^5 + 2^5 + 3^5 + 4^5 + \ldots + k^5 \).

**Solution:**

(a) We proceed by induction. For \( n = 0 \) the formula is trivial. Supposing that the formula holds for \( n \), we have \( B_{n+1}(x) = \int B_n(x)\,dx + B_{n+1}(0) = \frac{1}{(n+1)!} \sum_{k=0}^{n+1} \binom{n+1}{k} b_k x^{n+1-k} \).
10. POLYNOMIALS

(b) Remark first that \( 0 = \int_0^1 B_n(x)\,dx = B_{n+1}(1) - B_{n+1}(0) \). Using part (a) this becomes:
\[
\sum_{k=0}^{n+1} \binom{n+1}{k} b_k = b_{n+1},
\]
and this equation can be arranged as
\[
b_n = -\frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k} b_k.
\]

(c) We show by induction that \( B_n(x) = (-1)^n B_n(1-x) \). For \( n = 0 \), the formula is trivial. Suppose it is satisfied for \( n - 1 \). Denote \( C_n(x) = (-1)^n B_n(1-x) \). Then
\[
C_n(x) = (-1)^{n+1} B_n'(1 - x) = (-1)^n B_n(1-x) = B_{n-1}(x).
\]
Also a substitution proves
\[
\int_0^1 C_n(x)\,dx = \int_0^1 (-1)^n B_n(1-x)\,dx = (-1)^n \int_0^1 B_n(x)\,dx = 0.
\]
The polynomial \( C_n(x) \) satisfies the conditions which define \( B_n(x) \), therefore \( C_n(x) = B_n(x) \). Taking \( x = 0 \) in the equation \( B_{2n+1}(x) = -B_{2n+1}(1-x) \) gives \( B_{2n+1}(0) = -B_{2n+1}(1) = B_{2n+1}(0) \). Hence \( B_{2n+1} = 0 \).

(d) For \( n = 0 \) the equation \( B_1(x+1) - B_1(x) = 1 \), can easily be verified. Suppose that \( B_{n+1}(x+1) - B_{n+1}(x) = \frac{x^{n+1}}{(n+1)!} \). Then \( B_{n+1}(x+1) - B_{n+1}(x) = B_{n+1}(x+1) - B_{n+1}(1) + B_{n+1}(1) - B_{n+1}(x) = B_{n+1}(x+1) - B_{n+1}(1) + B_{n+1}(0) = \int_0^1 (B_n(t+1) - B_n(t))\,dt = \int_0^1 \frac{t^{n-1}}{(n-1)!} \,dt \).

(e) Take successively \( x = 0, 1, 2, \ldots, k \) in part (d).

(f) Using part (b) we calculate \( b_0 = 1, b_1 = -\frac{1}{2}, b_2 = \frac{1}{6}, b_3 = 0, b_4 = -\frac{1}{30}, b_5 = 0, b_6 = \frac{1}{42} \). The first Bernoulli polynomials are \( B_0(x) = 1, B_1(x) = x - \frac{1}{2} \),
\[
B_2(x) = \frac{x^2}{2} - \frac{x}{2} + \frac{1}{12}, B_3(x) = \frac{x^3}{6} - \frac{x^2}{4} + \frac{x}{12}, B_4(x) = \frac{x^4}{24} - \frac{x^3}{12} + \frac{x^2}{24} - \frac{1}{720},
\]
\[
B_5(x) = \frac{x^5}{120} + \frac{x^4}{48} + \frac{x^3}{72} - \frac{x}{720}.
\]

**Problem 10.121.** Let \( u_n \) be the unique positive root of \( x^n + x^{n-1} + \ldots + x^2 + x - 1 = 0 \). Find \( \lim_{n \to \infty} u_n \).

**Solution:** Multiplying the equation by \( x - 1 \) we see that \( u_n \) is the solution in \( 0, 1 \) of \( P(x) = x^{n+1} - 2x + 1 \). The derivative \( P'(x) = (n+1)x^n - 2 \) has the zero \( \sqrt{\frac{2}{n+1}} \) and this is a minimum point for \( P \) in the interval \( (0, \infty) \). Then
\[
P\left( \sqrt{\frac{2}{n+1}} \right) < f(1) = 0 \text{ and using also } f(0) = 1 \text{ we obtain } 0 < u_n < \sqrt{\frac{2}{n+1}}.
\]
As a consequence of these inequalities \( \lim_{n \to \infty} u_n^{n+1} = 0 \). But \( P(u_n) = 0 \) so \( \lim_{n \to \infty} u_n = \lim_{n \to \infty} \frac{1}{2} (u_n^{n+1} + 1) = \frac{1}{2} \).

**Problem 10.122.** Let \( P_n(x) = 1 + x + x^2 + \ldots + x^{n-1} \). For which \( n \) is the polynomial \( Q_n(x) = P_n(x^2) \) divisible by \( P_n(x) \)?

**Solution:** The polynomial \( Q_n \) is divisible by \( P_n \) for \( n \) odd. Indeed, \( P_n(-1) = 0 \) and \( Q_n(-1) = P_n(1) = n \) for \( n \) even, so \( Q_n \) is not divisible by \( P_n \). If \( n \) is odd, then

\[ P_n(x) = \frac{x^n - 1}{x - 1}, \quad \text{and} \quad Q_n(x) = \frac{x^{2n} - 1}{x^2 - 1} = \frac{x^n - 1}{x - 1} \cdot \frac{x^n + 1}{x + 1} = P_n(x)(x^{n-1} - x^{n-2} + \ldots + x^2 - x + 1). \]

**Problem 10.123.** Prove that \( \alpha = \sqrt{2} + \sqrt{3} \) is algebraic over \( \mathbb{Q} \), by explicitly finding a polynomial with coefficients in \( \mathbb{Q} \) of which \( \alpha \) is a root.

**Solution:** We have \((\alpha - \sqrt{3})^2 = 5\) and from this \((\alpha^2 - 2)^2 = 12\alpha^2\). Therefore the polynomial \( P(x) = x^4 - 16x^2 + 4 \) has \( \alpha \) as root.

**Problem 10.124.** If \( a_0, a_1, \ldots, a_n \) are all different, and \( b_0, b_1, \ldots, b_n \) are complex numbers, prove that there is a unique polynomial \( f \) of degree \( n \) with complex coefficients such that \( f(a_i) = b_i \), for \( 0 \leq i \leq n \).

**Solution:** The polynomial \( f \) is of the form \( f(x) = c_0 + c_1 x + \ldots + c_n x^n \). The system of \( n \)-equations \( f(a_i) = b_i \), \( 0 \leq i \leq n \), in the \( n \)-variables \( c_0, c_1, \ldots, c_n \) has the determinant \( V(a_0, a_1, \ldots, a_n) \neq 0 \) and therefore has an unique solution.

**Problem 10.125.** 1. Evaluate \( P_{n-1}(1) \), where \( P_{n-1}(x) = \frac{x^n - 1}{x - 1} \).

2. Consider a circle of radius 1, and let \( Q_1, Q_2, \ldots, Q_n \) be the vertices of a regular \( n \)-gon inscribed in the circle. Join \( Q_1 \) to \( Q_2, Q_3, \ldots, Q_n \) by segments of a straight line. You obtain \( (n - 1) \) segments of lengths \( \lambda_2, \lambda_3, \ldots, \lambda_n \). Show that \( \prod_{i=2}^{n} \lambda_i = n \).

**Solution:** 1. \( P_{n-1}(x) = x^{n-1} + x^{n-2} + \ldots + x + 1 \), so \( P_{n-1}(1) = n \).

2. Consider a system of coordinates such that the point \( Q_1 \) has the coordinates \((1,0)\). If \( O \) is the center of the circle, the angle \( \angle Q_1 O Q_k \) is \( \frac{2(k-1)\pi}{n} \), so the point \( Q_k \) has the coordinates \( z_k = \cos \frac{2(k-1)\pi}{n} + i \sin \frac{2(k-1)\pi}{n} \). The length \( \lambda_k \) is given then by \( |1 - z_k| \). Since \( P_{n-1}(x) = \prod_{k=2}^{n} (x - z_k) \), the product \( \prod_{i=2}^{n} \lambda_i \) is \( |P_{n-1}(1)| = n \).

**Problem 10.126.** Simplify the polynomial \( P(x) = (x-a)(x-b)(x-c)\ldots(x-z) \)

**Solution:** One of the factors is \( x-x = 0 \), so \( P(x) = 0 \).

**Problem 10.127.** For a real number let \( S(x) \) denote the sequence of integers \([x], [2x], [3x], \ldots\). Prove that there are distinct real solutions \( \alpha \) and \( \beta \) of the equation \( P(x) = x^3 - 10x^2 + 29x - 25 = 0 \) such that infinitely many positive integers appear in both \( S(\alpha) \) and \( S(\beta) \).

**Solution:** It suffices to show that there are two distinct positive solutions \( \alpha \) and \( \beta \) and this is a consequence of \( P(1) = -5 \), \( P(2) = 1 \), \( P(3) = -2 \).

**Problem 10.128.** Let \( f(x) \), \( g(x) \), \( h(x) \) be polynomials with real coefficients such that \( f^2(x) = g^2(x) + h^2(x) \). Assume that \( 0 < \deg(g) < n \), where \( n \) is the degree of \( f(x) \). Show that \( f(x) \) has fewer than \( n \) real roots.

**Solution:** Suppose that \( f(x) \) has \( n \) real roots. Then they are also roots for \( g(x) \) which contradicts the fact that \( 0 < \deg(g) < n \).
10. POLYNOMIALS

Problem 10.129. Define inductively a sequence of polynomials by \( P_1(x) = x^2 - 2 \) and \( P_{n+1}(x) = P_n(P_n(x)) \). Prove that for each \( n \geq 1 \) the polynomial \( P_n(x) \) has \( 2^n \) distinct real roots.

**Solution**: It is not difficult to see that \( P_n(x) \) has the degree \( 2^n \) and also that \( P_{n+1}(x) = P_n(P_1(x)) \). We prove by induction that the \( 2^n \) roots of \( P_n(x) \) are \( 2 \cos \frac{(2k + 1)\pi}{2^{n+1}} \), \( k = 0, 1, 2, \ldots, 2^n - 1 \). Indeed, \( P_1(x) \) has the roots \( 2 \cos \frac{\pi}{4} \) and \( 2 \cos \frac{3\pi}{4} \). We suppose that the roots of \( P_n(x) \) are those indicated. Then for \( k = 0, 1, \ldots, 2^{n+1} - 1 \) the computation \( P_{n+1}(2 \cos \frac{(2k + 1)\pi}{2^{n+2}}) = P_n(4 \cos^2 \frac{(2k + 1)\pi}{2^{n+1}} - 2) = P_n(2 \cos \frac{(2k + 1)\pi}{2^{n+1}}) = 0 \), shows that the roots of \( P_{n+1}(x) \) are \( 2 \cos \frac{(2k + 1)\pi}{2^{n+2}} \), \( k = 0, 1, 2, \ldots, 2^{n+1} - 1 \).

Problem 10.130. Let \( P(x) \) be a nonzero polynomial with real coefficients such that \( P(a) \geq 0 \) for all real numbers \( a \). Prove there exists a polynomial \( Q(x) \) with real coefficients with the following two properties:

(a) \( P(a) \geq Q^2(a) \) for all real \( a \)

(b) \( \deg(P(x) - Q^2(x)) < \deg P(x) \)

**Solution**: All real roots of \( P \) have even multiplicity and if \( a + ib \) is a complex root, then \( a - ib \) is also a root of \( P(x) \) with the same multiplicity like \( a + ib \). Then the polynomial can be written

\[
P(x) = (x - x_1)^{2n_1}(x - x_2)^{2n_2} \cdots (x - x_p)^{2n_p}[(x - a_1)^2 + b_1^2] \cdots [(x - a_q)^2 + b_q^2]
\]

and \( Q(x) = (x - x_1)^{n_1}(x - x_2)^{n_2} \cdots (x - x_p)^{n_p}(x - a_1) \cdots (x - a_q) \) satisfies the two properties.

Problem 10.131. Let \( P(x) \) be a polynomial of degree \( n \) and \( Q(x) \) a quadratic polynomial such that \( P(x) = Q(x)P''(x) \). Show that if \( P(x) \) has at least two distinct roots then it must have \( n \) distinct roots. \( \{P1999\} \)

**Solution**: Suppose \( P \) has a root \( x = a \) with multiplicity at least 2, which means \( P(a) = 0 \) and \( P'(a) = 0 \). Identifying the coefficients which gives the order one can see that \( Q(x) = \frac{1}{n(n-1)} x^2 + \alpha x + \beta \), so \( Q''(x) = \frac{1}{n(n-1)} \). Taking the \( k \)-th derivative of the equality \( P(x) = Q(x)P''(x) \) we obtain

\[
(12.3) \quad P^{(k)}(x) = \frac{k(k-1)}{2} \frac{1}{n(n-1)} P^{(k)}(x) + kQ'(x)P^{(k+1)}(x) + Q(x)P^{(k+2)}(x)
\]

We distinguish the following cases:

a) \( Q(a) = Q'(a) = 0 \)

In this case, for \( x = a \), 12.3 becomes \( P^{(k)}(a) = \frac{k(k-1)}{2n(n-1)} P^{(k)}(a), \) so \( P^{(k)}(a) = 0 \) for any \( k \). Then \( P(x) = c(x-a)^n \), which contradicts the hypothesis that \( P \) has 2 distinct solutions.

b) \( Q(a) = 0, \ Q'(a) \neq 0 \)
For $x = a$, 12.3 becomes $P^{(k)}(a) = \frac{k(k-1)}{2n(n-1)} P^{(k)}(a) + kQ'(a)P^{(k+1)}(a)$. An easy induction based on this equality shows that $P^{(k)}(a) = 0$ for any $k$. Contradiction.

c) $Q(a) \neq 0$

From 12.3 we get

$$P^{(k+2)}(a) = \frac{1}{Q(a)} \left( P^{(k)}(a) - \frac{k(k-1)}{2n(n-1)} P^{(k)}(a) - kQ'(a)P^{(k+1)}(a) \right)$$

which by induction yields $P^{(k)}(a) = 0$ for any $k$. Contradiction.

Problem 10.132. Prove that there is a constant $C$ such that, if $p(x)$ is a polynomial of degree 1999, then $|p(0)| \leq C \int_{-1}^{1} |p(x)| \, dx$. [P1999]

Solution: The maps $p \mapsto \int_{-1}^{1} |p(x)| \, dx$ and $p \mapsto \sup \{|p(x)|, x \in [-1,1]\}$ are two norms on the finite dimensional vector space of polynomials of degree $\leq 1999$. But all the norms on a finite dimensional vector space are equivalent. So, in particular there is a constant $C > 0$ such that

$$\int_{-1}^{1} |p(x)| \, dx \geq \frac{1}{C} \sup \{|p(x)|, x \in [-1,1]\} \geq \frac{1}{C} |p(0)|$$

Problem 10.133. Find polynomials $f(x), g(x)$, and $h(x)$, if they exist, such that for all $x$,

$$|f(x)| - |g(x)| + h(x) = \begin{cases} -1 & \text{if } x < -1 \\ 3x + 2 & \text{if } -1 \leq x \leq 0 \\ -2x + 2 & \text{if } x > 0. \end{cases}$$

[P1999]

Solution: We will look for $f$ and $g$ first order polynomials. Let $\phi(x)$ be the function defined by the right side of the above equation. In the point $x = -1$ the function $\phi$ has a convex corner and the change of slope $\phi'(-1) - \phi'(1) = 3$. The function $-|g|$ can have only concave corners, $h$ doesn’t have any corners, and $f$ can only have convex corners. Therefore $f(-1) = 0$ and also $|f'|(-1) - |f'|(-1) = 3$ (since $|g|$ and $h$ don’t have change of slope in $x = -1$). The only first order polynomial satisfying these conditions is $f(x) = \frac{3}{2}x + \frac{3}{2}$.

In the point $x = 0$ the function $\phi$ has a concave corner and change of slope $\phi'(-1) - \phi'(1) = 3$. The corresponding conditions on $|g|$ yield $g(x) = \frac{5}{2}x$. Replacing $f$ and $g$ in the equation we obtain $h(x) = -x + \frac{1}{2}$ for $x \leq -1$, and this $h$ satisfies the equation also for the other values of $x$.

Problem 10.134. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$; $f_n(0) = 0$ for $n \geq 1$, and

$$\frac{d}{dx} f_{n+1}(x) = (n+1)f_n(x + 1)$$
for $n \geq 0$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes. [P1985]

**Solution :** We will prove that $f_n(x) = x(x + n)^{n-1}$ for any $n \geq 1$. For $n = 1$ we have $f_1(x) = x$. We suppose $f_k(x) = x(x + k)^{k-1}$. Then $\frac{d}{dx}f_{k+1}(x) = (k + 1)(x + 1)(x + k + 1)^{k-1} = (k + 1)(x + k + 1)^k - k(k + 1)(x + k + 1)^{k-1}$. Therefore, $f_{k+1}(x) = (x+k+1)^{k+1} - (k+1)(x+k+1)^k = x(x+k+1)^k$. Hence $f_{100}(1) = 101^{99}$.

**Problem 10.135.** Let $a_1, a_2, \ldots, a_n$ be real numbers, and let $b_1, b_2, \ldots, b_n$ be distinct positive integers. Suppose that there is a polynomial $f(x)$ satisfying the identity

$$(1 - x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$ 

Find a simple expression (not involving any sums) for $f(1)$ in terms of $b_1, b_2, \ldots, b_n$ and $n$ (but independent of $a_1, a_2, \ldots, a_n$). [P1986]

**Solution :** Taking the $k$-th derivative of the equation, with $k = 0, 1, 2, \ldots, n$, for $x = 1$ we get the following system:

$$
\begin{align*}
\sum_{i=1}^{n} a_i &= -1 \\
\sum_{i=1}^{n} a_i b_i &= 0 \\
\sum_{i=1}^{n} a_i b_i (b_i - 1) &= 0 \\
&\vdots \\
\sum_{i=1}^{n} a_i b_i (b_i - 1) \ldots (b_i - n + 2) &= 0 \\
\sum_{i=1}^{n} a_i b_i (b_i - 1) \ldots (b_i - n + 2) (b_i - n + 1) &= (-1)^n f(1)
\end{align*}
$$

We add the second equation to the third one, then we add 3 times the third equation and we subtract 2 times the second one from the fourth equation, and in general at the $k$-th equation we add and subtract a linear combination of the first $k - 1$ equations to get
12. VARIA

\[
\begin{align*}
\sum_{i=1}^{n} a_i &= -1 \\
\sum_{i=1}^{n} a_i b_i &= 0 \\
\sum_{i=1}^{n} a_i b_i^2 &= 0 \\
& \vdots \\
\sum_{i=1}^{n} a_i b_i^{n-1} &= 0 \\
\sum_{i=1}^{n} a_i b_i^n &= (-1)^n n! f(1)
\end{align*}
\]

(12.4)

The determinant of the system consisting in the first \( n \) equations is the non-zero Vandermonde determinant

\[
\Delta = \begin{vmatrix}
1 & 1 & \ldots & 1 \\
b_1 & b_2 & \ldots & b_n \\
\vdots & \vdots & \ddots & \vdots \\
1 & b_2 & \ldots & b_n
\end{vmatrix}
\]

By Cramer’s formula,

\[
a_1 = \frac{1}{\Delta} \begin{vmatrix}
-1 & 1 & \ldots & 1 \\
0 & b_2 & \ldots & b_n \\
\vdots & \vdots & \ddots & \vdots \\
0 & b_n & \ldots & b_2
\end{vmatrix} = -\frac{1}{\Delta} \begin{vmatrix}
b_2 & \ldots & b_n \\
\vdots & \ddots & \vdots \\
b_2^{n-1} & \ldots & b_n^{n-1}
\end{vmatrix}
\]

(12.5)

The determinant of the system formed by the last \( n \) equations of the system 12.4 is \( \Delta_1 = \begin{vmatrix}
b_1 & b_2 & \ldots & b_n \\
\vdots & \vdots & \ddots & \vdots \\
b_1^n & b_2^n & \ldots & b_n^n
\end{vmatrix} = b_1 b_2 \ldots b_n \Delta \), so

\[
a_1 = \frac{1}{b_1 b_2 \ldots b_n \Delta} \begin{vmatrix}
0 & b_2 & \ldots & b_n \\
\vdots & \ddots & \vdots & \vdots \\
(1)^n n! f(1) & b_2^n & \ldots & b_n^n
\end{vmatrix} = \frac{(-1)^{n+1} (-1)^n n! f(1)}{b_1 b_2 \ldots b_n \Delta}
\]

The last equation together with 12.5 yield \( f(1) = \frac{b_1 b_2 \ldots b_n}{n!} \).

**Problem 10.136.** Find all real polynomials \( p(x) \) of degree \( n \geq 2 \) for which there exist real numbers \( r_1 < r_2 < \cdots < r_n \) such that

1. \( p(r_i) = 0 \), \( i = 1, 2, \ldots, n \), and
2. \( p'\left(\frac{r_i + r_{i+1}}{2}\right) = 0 \), \( i = 1, 2, \ldots, n - 1 \),
where \( p'(x) \) denotes the derivative of \( p(x) \). [P1991]

**Solution** : The polynomial \( p(x) \) must be of the form \( a(x - r_1)(x - r_2)...(x - r_n) \). We will show that the problem has a solution only for \( n = 2 \). Indeed, if \( n \geq 3 \), then \( p(x) = (x - r_1)(x - r_2)q(x) \), with \( q(x) = a(x - r_3)...(x - r_n) \). Then
\[
\begin{align*}
\frac{p'(x)}{p(x)} &= -\left(\frac{r_1 - r_2}{2}\right)^2 q'(\frac{r_1 + r_2}{2}).\end{align*}
\]
Since \( q'(x) = q(x) \sum_{k \geq 3} \frac{1}{x - r_k} \) it is easy to that \( q'(\frac{r_1 + r_2}{2}) \) cannot be 0. In the case \( n = 2 \), \( p(x) = (x - r_1)(x - r_2) \) satisfies the requirements.

**Problem 10.137.** For each integer \( m \), consider the polynomial
\[
P_m(x) = x^4 - (2m + 4)x^2 + (m - 2)^2.
\]
For what values of \( m \) is \( P_m(x) \) the product of two non-constant polynomials with integer coefficients? [P2001]

**Solution** : The zeros of \( P_m \) are \( x_1 = \sqrt{m} - \sqrt{2}, x_2 = \sqrt{m} + \sqrt{2}, x_3 = -\sqrt{m} + \sqrt{2} \) and \( x_4 = -\sqrt{m} - \sqrt{2} \).

If one of the factors of \( P_m \) is of the first order, the dominant coefficient of it must be \( \pm 1 \), so \( P_m \) has an integer zero and this can be only \( x_1 \) for \( m = 2 \).

If \( P_m = A_m B_m \), with \( A_m \) and \( B_m \) polynomials with integer coefficients we distinguish the following situations.

a) \( x_1, x_2 \) are the zeros of \( A_m \) and \( x_3, x_4 \) the zeros of \( B_m \). Then \( x_1 + x_2 = 2\sqrt{m} \) must be an integer, so \( m = k^2 \) for \( k \in \mathbb{Z} \). We have \( P_{k^2}(x) = (x^2 - 2kx + k^2 - 2)(x^2 + 2kx + k^2 - 2) \).

b) \( x_1, x_3 \) are the zeros of \( A_m \) and \( x_2, x_4 \) the zeros of \( B_m \). Then \( x_1x_3 = -m - 2 - 2\sqrt{m} \) must be an integer, so \( m = 2k^2 \). In this case \( P_{2k^2}(x) = (x^2 - 2k^2 - 4k - 2)(x^2 - 2k^2 + 4k - 2) \).

c) \( x_1, x_4 \) are the zeros of \( A_m \) and \( x_2, x_3 \) the zeros of \( B_m \). Then \( x_1 + x_4 = -2\sqrt{2} \) must be an integer, contradiction.

**Problem 10.138.** Find all \( c \) such that the graph of the function \( x^4 + 9x^3 + cx^2 + ax + b \) meets some line in four distinct points. [P1994]

**Solution** : We are looking for the values of \( c \) such that there are \( a, b, \alpha, \beta \) with the property that the equation \( x^4 + 9x^3 + cx^2 + ax + b = \alpha x + \beta \) has 4 distinct zeros. These are the same values like those for which the polynomial \( f(x) = x^4 + 9x^3 + cx^2 + ax + b \) has 4 distinct zeros. For such \( c \) the equation \( f'(x) = 0 \) has 3 distinct zeros and \( f''(x) = 12x^2 + 54x + 2c = 0 \) has 2 distinct zeros. These last condition means \( c < \frac{243}{8} \) and we will show that these are the desired values of \( c \). Let \( x_1 \) and \( x_2 \) be the two distinct zeros of \( f'(x) = 0 \). The function \( f'(x) \) is increasing on \( (-\infty, x_1] \cup [x_2, \infty) \) and decreasing on \( (x_1, x_2) \). The equation \( f'(x) = 0 \) has 3 distinct zeros if \( f'(x_1) > 0 \) and \( f'(x_2) < 0 \) and these two conditions holds if \( a < -4x_1^3 - 27x_1^2 - 2cx_1 < a < -4x_2^3 - 27x_2^2 - 2cx_2 \). Let \( y_1, y_2, y_3 \) be the distinct zeros of \( f''(x) = 0 \). The the function \( f(x) \) is decreasing on \( (-\infty, y_1) \cup (y_2, y_3) \) and increasing on \( [y_1, y_2) \cup [y_3, \infty) \). The equation \( f(x) = 0 \) has 4 distinct zeros if \( f(y_1) < 0, f(y_2) > 0 \) and \( f(y_3) < 0 \). These conditions are satisfied if we choose the \( b \) such that \( -y_2^2 - 9y_2^3 - cy_2^2 - ay_2 < b < \min(-y_1^2 - 9y_1^3 - cy_1^2 - ay_1, -y_3^3 - 9y_3^3 - cy_3^2 - ay_3) \)
Problem 10.139. Let \( k \) be the smallest positive integer for which there exist distinct integers \( m_1, m_2, m_3, m_4, m_5 \) such that the polynomial
\[
p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)
\]
has exactly \( k \) nonzero coefficients. Find a set of integers \( m_1, m_2, m_3, m_4, m_5 \) for which this minimum \( k \) is achieved. [P1985]

Solution: We prove \( k \geq 3 \). Indeed if \( k \leq 2 \), then there are \( i \in \{0, 1, 2, 3, 4\} \) and \( a \) integer such that \( p(x) = x^5 - ax^4 \). If \( i \geq 2 \) then the multiplicity of the root 0 of \( p(x) \) is greater than 1. If \( i = 1 \) then \( p(x) \) has at least two zeros which are not real, and if \( i = 0 \) then \( p(x) \) has 4 zeros which are not real. Contradiction. The example \( p(x) = x(x - 1)(x + 1)(x - 2)(x + 2) = x^5 - 5x^3 + 4x \) shows that \( k = 3 \).

Problem 10.140. Find, with explanation, the maximum value of \( f(x) = x^3 - 3x \) on the set of all real numbers \( x \) satisfying \( x^4 + 36 \leq 13x^2 \). [P1986]

Solution: The inequality \( x^4 + 36 \leq 13x^2 \) is equivalent with \( (x^2 - 4)(x^2 - 9) \leq 0 \) and is satisfied for \( x \in [-3, -2] \cup [2, 3] \). The function \( f \) is increasing on these two intervals, so the maximum is \( \max(f(-2), f(3)) = 18 \).

Problem 10.141. Prove that there are only a finite number of possibilities for the ordered triple \( T = (x - y, y - z, z - x) \), where \( x, y, z \) are complex numbers satisfying the simultaneous equations
\[
x(x - 1) + 2yz = y(y - 1) + 2zx = z(z - 1) + 2xy,
\]
and list all such triples \( T \). [P1986]

Solution: With the notations \( u = x - y, v = y - z \) we have \( z - x = -u - v \). The numbers \( x, y, z \) are satisfying the system \( x^2 - x + 2yz = A, y^2 - y + 2zx = A, z^2 - z + 2xy = A \). Subtracting the second equation from the first one and the third from the second we get the system
\[
\begin{align*}
u(u + 2v - 1) &= 0 \\
v(2u + v + 1) &= 0
\end{align*}
\]
which has the solutions \( (0,0), (0,-1), (1,0), (-1,1) \). The answer is therefore \( (0,0,0), (0,-1,1), (1,0,-1), (-1,1,0) \).

Problem 10.142. Let \( p(x) \) be a polynomial that is nonnegative for all real \( x \). Prove that for some \( k \), there are polynomials \( f_1(x), \ldots, f_k(x) \) such that \( p(x) = \sum_{j=1}^k (f_j(x))^2 \). [P1999]

Solution: First factor \( p(x) = q(x)r(x) \), where \( q \) has all real roots and \( r \) has all complex roots. Notice that each root of \( q \) has even multiplicity, otherwise \( p \) would have a sign change at that root. Thus \( q(x) \) has a square root \( s(x) \).

Now write \( r(x) = \prod_{j=1}^k (x - a_j)(x - \overline{a_j}) \) (possible because \( r \) has roots in complex conjugate pairs). Write \( \prod_{j=1}^k (x - a_j) = t(x) + iu(x) \) with \( t, x \) having real coefficients. Then for \( x \) real,
\[
p(x) = q(x)r(x) = s(x)^2(t(x) + iu(x))(t(x) + iu(x)) = (s(x)t(x))^2 + (s(x)u(x))^2.
\]
(Alternatively, one can factor \( r(x) \) as a product of quadratic polynomials with real coefficients, write each as a sum of squares, then multiply together to get a sum of many squares.)
Second solution: We proceed by induction on the degree of $p$, with base case where $p$ has degree 0. As in the first solution, we may reduce to a smaller degree in case $p$ has any real roots, so assume it has none. Then $p(x) > 0$ for all real $x$, and since $p(x) \to \infty$ for $x \to \pm\infty$, $p$ has a minimum value $c$. Now $p(x) - c$ has real roots, so as above, we deduce that $p(x) - c$ is a sum of squares. Now add one more square, namely $(\sqrt{c})^2$, to get $p(x)$ as a sum of squares.

Problem 10.143. Determine all the zeros of the polynomial

$$P(x) = 1 + \sum_{k=1}^{n} \frac{x(x + 1)...(x + k - 1)}{k!}.$$

Problem 10.144. Let $P$ be a cubic polynomial such that $P(1) = 2$, $P(2) = 4$, $P(3) = 8$ and $P(4) = 16$. Determine $P(5)$. Generalize.
CHAPTER 11

The geometry of the triangle

Problem 11.1. If in a triangle $ABC$ we have $a \geq b$, then $am_a^n \leq bm_b^n$, for any $n \geq 1$.

Solution: For $n = 1$ we prove by direct computation. Then multiply by $m_a^{n-1} \leq m_b^{n-1}$.

Problem 11.2. Prove there is no real constant $k$ such that for any triangle $ABC$ we have $a^2 + b^2 + c^2 \leq kS$, or $(a + b + c)^2 \leq kS$.

Solution: Take the degenerated triangle with $a = b + c$. Then $S = 0$.

Problem 11.3. Find a necessary and sufficient condition on the constants $h, k$ such that the inequality $a^2 + b^2 + c^2 \geq hr + kS$, holds for any triangle $ABC$.

Solution: Taking the equilateral triangle $ABC$ we see $k + \frac{h}{3\sqrt{3}} \leq \frac{a^2}{4\sqrt{3}}$ is a necessary condition.

We prove this is also a sufficient condition.

\[
kS + hr^2 \leq k \frac{a^2 + b^2 + c^2}{4\sqrt{3}} + h \frac{a^2 + b^2 + c^2}{36} \leq a^2 + b^2 + c^2.
\]

Problem 11.4. Prove that in any triangle the center of the circle inscribed is in the interior of the triangle determined by the midpoints of the sides.

Solution: This is equivalent with $r < \frac{1}{2} \leq \min\{h_a, h_b, h_c\}$. For example $r = \frac{S}{p} = \frac{ab_2}{2p} < \frac{h_a}{2}$.

Problem 11.5. Prove that in a triangle $ABC$, with a right angle in $A$, the inequality $\frac{ap}{bc} \geq 1 + \sqrt{2}$ holds.

Solution: $\frac{ap}{bc} = \frac{b^2 + c^2}{2bc} + \frac{(b + c)\sqrt{b^2 + c^2}}{2bc} \geq 1 + \frac{2\sqrt{bc}\sqrt{2bc}}{2bc} = 1 + \sqrt{2}$.

Problem 11.6. If in a triangle $b^2 - c^2 = 2a^2$, then $A \leq \frac{\pi}{6}$.

Solution: The condition is equivalent with $2 \sin A = \sin(B - C)$. It follows $\sin A \leq \frac{1}{2}$, or $A \leq \frac{\pi}{6}$.

Problem 11.7. If $A, B, C$ are the angles of a triangle then for any real $x$,

\[
x(x - 1) \sin C + x \sin B \sin c + \sin C > 0
\]
Solution: The inequality is equivalent with \( f(x) = x^2\sin(A + B) + x(\sin A - 
\sin B - \sin(A + B)) + \sin B > 0 \), for all real \( x \). But the discriminant of \( f \) is 
\[ \Delta = -8 \sin \frac{B}{2} \cos \frac{A+B}{2} \left( \sin^2 \frac{A}{2} \sin \frac{B}{2} \cos \frac{A+B}{2} + \sin \frac{A}{2} \right) < 0. \]

Problem 11.8. If \( a, b, c \) are the lengths of the sides of a triangle, then 
\[ a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \geq 0 \]

Solution: Suppose \( a \geq b \geq c \). We study the function \( f(a) = a^2b(a - b) + b^2c(b - c) + c^2a(c - a) \) on the interval \( [b, b + c) \). We have \( f'(a) = 2ab - 2ac^2 + c^3 + 2ab(a - b) \geq c(a - c)^2 + 2ab(a - b) \geq 0 \), so \( f \) is an increasing function. Hence \( f(a) \geq f(b) = bc(b - c)^2 \geq 0 \). The equality holds when \( a = b = c \).

Problem 11.9. Prove that \( a^2 + b^2 + c^2 \geq 4\sqrt{3}S \).

Solution: Taking the square, using Heron’s formula, and arranging convenably we get \( 2(a^2 - b^2)^2 + 2(b^2 - c^2)^2 + 2(c^2 - a^2)^2 \geq 0 \).

Second solution: Adding the inequalities of the form \( ab_a \leq am_a \) we have \( 6S \leq am_a + bm_b + cm_c \). Also \( (am_a + bm_b + cm_c)^2 \leq (a^2 + b^2 + c^2)(m_a^2 + m_b^2 + m_c^2) = \frac{3}{4}(a^2 + b^2 + c^2)^2 \).

Problem 11.10. Prove that the sums of squares of the lengths of the sides of a convex polygon circumscribed to a circle is smaller or equal than \( 9R^2 \).

Problem 11.11. Given a point \((a, b)\) with \( 0 < b < a \), determine the minimum perimeter of a triangle with one vertex at \((a, b)\), one on the \( x \)-axis, and one on the line \( y = x \). You may assume that a triangle of minimum perimeter exists. [P1998]

Solution: Let \( E(a, -b) \) (respectively \( F(b, a) \)) be the symmetrical of the point \( A(a, b) \) with respect to the \( x \)-axis (respectively the line \( y = x \)). Consider \( B \) a point on the \( x \)-axis and \( C \) a point on the line \( y = x \). Then \( AC = CF \) and \( AB = BE \), so the perimeter of the triangle \( ABC \) is equal with the sum of the lengths of the segments \( BE, BC \) and \( CF \). But this is minimum when \( B \) is the intersection of \( EF \) with the \( x \)-axis, and \( C \) is the intersection of \( EF \) with the line \( y = x \). The minimum is therefore the length of the segment \( EF \), i.e. \( \sqrt{2(a^2 + b^2)} \).

Problem 11.12. The length of two altitudes of a triangle are \( h \) and \( k \). Find an upper and a lower bound for the length of the third altitude in terms of \( h \) and \( k \).

Solution: Let \( l \) be the length of the third altitude, \( a, b, c \) the lengths of the sides, and \( S \) the area of the triangle. Replacing \( a = 2S/h, k = 2S/k, c = 2S/l \) in the triangle inequality \( |a-b| < c < a+b \), gives \( \frac{hk}{h+k} < l < \frac{hk}{|h-k|} \), with the convention \( h^2 \_0 = \infty \).

Problem 11.13. If \( S \) is the area of triangle \( ABC \) then \( abc \geq \frac{8S^{3/2}}{3^{3/4}} \) with equality for the equilateral triangle.
Solution: Since \( S^3 = \frac{ab \sin C \cdot bc \sin A \cdot ca \sin B}{2} \) the inequality is equivalent with \( \frac{3\sqrt{3}}{2} \geq \sin A \sin B \sin C \). But this is a consequence of Jensen’s inequality

\[
f\left(\frac{A + B + C}{3}\right) \geq \frac{f(A) + f(B) + f(C)}{3}
\]

applied to the concave function \( f(x) = \sin x \).

Problem 11.14. Let \( ABC \) be a triangle and \( a, b, c \) the lengths of the sides, \( S \) the area, \( p \) the half-perimeter, \( r, R \) the radius of the circle inscribed, respectively circumscribed, \( m_a, m_b, m_c \) the medians opposed respectively to \( a, b, c \), and \( l_a, l_b, l_c \) the bisectors opposed respectively to \( a, b, c \). Then

1. \( a^2 + b^2 + c^2 \geq 36r^2 \)
2. \( ab + bc + ca \geq 4S\sqrt{3} \geq 36r^2 \)
3. \( p \geq 3\sqrt{3}r \)
4. \( S \geq 3\sqrt{3}r^2 \)
5. \( p^2 \geq 3\sqrt{3}S \)
6. \( R \geq \frac{2\sqrt{3}}{9} \geq 2r \)
7. If \( a \geq b \geq c \) then \( am_a^n \leq bm_b^n \leq cm_c^n \), for any \( n \in \mathbb{N}^* \).
8. If \( a \geq b \geq c \) then
   (a) \( l_a \leq b \leq l_c \)
   (b) \( a l_a^2 \leq b l_b^2 \leq c l_c^2 \)
   (c) \( bl_b \geq cl_c \)

Solution: 1. Denoting by \( l_a \) the bisector opposed to \( a \) it is known that \( l_a = \frac{2\sqrt{bc}}{b + c} \sqrt{p(p - a)} \). Then \( l_a^2 \leq p(p - a) \). Adding this with the similar inequalities we obtain \( l_a^2 + l_b^2 + l_c^2 \leq p^2 \). From \( l_a \geq h_a \), \( a l_a \geq 2S \) and adding it with the other similar inequalities \( a l_a + b l_b + c l_c \geq 6S \). Combining with Schwartz’s inequality one gets \( p^2(a^2 + b^2 + c^2) \geq (a^2 + b^2 + c^2)(l_a^2 + l_b^2 + l_c^2) \geq 36S^2 \). The identity \( S = pr \) finishes the proof.

2. Using Heron’s formula and taking the square, one can write the inequality under the form

\[
(a - b)^2 \left[ \frac{3}{2}(a + b)^2 - c^2 \right] + (b - c)^2 \left[ \frac{3}{2}(b + c)^2 - a^2 \right] + (c - a)^2 \left[ \frac{3}{2}(c + a)^2 - b^2 \right] \geq 0
\]

8a. The inequality to prove is \( \frac{2\sqrt{bc}}{b + c} \sqrt{p(p - a)} \leq \frac{2\sqrt{ac}}{a + c} \sqrt{p(p - b)} \) which is equivalent with \( \frac{\sqrt{b(p - a)}}{b + c} \leq \frac{a(p - b)}{a + c} \) or \( \left( \frac{a + c}{b + c} \right)^2 \leq \frac{a(a - b + c)}{b(b - a + c)} \). It suffices to multiply the following inequalities \( \frac{a}{b} \geq \frac{a + c}{b + c} \) and \( \frac{a - b + c}{b - a + c} \geq \frac{a + c}{b + c} \).

Problem 11.15. Find two non-congruent similar triangles with sides of integral length having the lengths of two sides of one triangle equal to the lengths of two sides of the other.

Solution: Let \( a \leq b \leq c \) and \( a' \leq b' \leq c' \) be the sides of the two triangles, and \( b = a' \), \( c = b' \). The similarity of the triangles means there is a rational number \( k \) such
Prove that there are exactly three right-angled triangles whose sides are integers while the area is numerically equal to twice the perimeter.

Solution: The sides of the right triangle must be of the form $m^2 - n^2$, $2mn$ and $m^2 + n^2$ with $m$ and $n$ integers. The condition between the area and the perimeter is then $\frac{2mn \cdot (m^2 - n^2)}{2} = 2((m^2 - n^2) + 2mn + (m^2 + n^2))$, or $n(m - n) = 4$. The only possibilities are

- $n = 1$ and $m - n = 4$ which gives the triangle with the sides $(24, 10, 26)$
- $n = 2$ and $m - n = 2$ which gives the triangle with the sides $(12, 16, 20)$
- $n = 4$ and $m - n = 1$ or the triangle with the sides $(9, 40, 41)$.

Problem 11.17. Let $ABC$ be a triangle with sides $BC$, $CA$, $AB$ of lengths $a, b, c$, respectively. Let $D, E$ be the midpoints of the sides $AC$, $AB$, respectively. Prove that the median $BD$ is perpendicular to $CE$ if and only if $b^2 + c^2 = 5a^2$.

Solution: Let $M$ be the intersection point of $CE$ and $BD$. Then $BD$ is perpendicular on $CE$ if and only if $BM^2 + CM^2 = a^2$. But $CM^2 = \frac{4}{9} CE^2 = \frac{4}{9} \left(\frac{a^2 + b^2}{2} - \frac{c^2}{4}\right)$ and $BM^2 = \frac{4}{9} BD^2 = \frac{4}{9} \left(\frac{a^2 + c^2}{2} - \frac{b^2}{4}\right)$.

Problem 11.18. Let $ABC$ be an acute angled triangle, and let $D$ be a point on the line $BC$ for which $AD$ is perpendicular to $BC$. Let $P$ be a point on the line segment $AD$. The lines $BP$ and $CP$ intersect $AC$ and $AB$ at $E$ and $F$ respectively. Prove that the line $AD$ bisects the angle $EDF$.

Problem 11.19. Let $ABC$ be a triangle with the right angle in $A$ and $D, E$ two points on the segment $BC$ such that $CE = ED = DB$. What are values that the quotient $\frac{AD}{AE}$ can take?

Problem 11.20. Find a necessary and sufficient condition on the positive real constants $h, k$ such that the inequality $a^2 + b^2 + c^2 \geq kS + hr^2$ holds for any triangle. The notations are the usual ones.

Solution: For an equilateral triangle the inequality becomes $k + \frac{h}{3\sqrt{3}} \leq 4\sqrt{3}$.

Let us show this condition is also sufficient. Indeed, $kS + hr^2 \leq k \frac{a^2 + b^2 + c^2}{4\sqrt{3}} + h \frac{a^2 + b^2 + c^2}{36} = \frac{a^2 + b^2 + c^2}{4\sqrt{3}} \left(k + \frac{h}{3\sqrt{3}}\right) \leq a^2 + b^2 + c^2$. 

\[ a = \frac{b'}{a} = \frac{c'}{a} = k. \] Then \( a' = ka, b' = kb = ka' = k^2a, c' = kc = kb' = k^3a. \) The numbers \( a', b', c' \) are sides of a triangle if and only if \( k + k^2 > k^3, k + k^3 > k^2 \) and \( k^2 + k^3 > k \). This inequalities are satisfied for example for \( k = 3/2 \). One can take then \( a = 8 \) and we see that the triangles with the sides \((8, 12, 18)\) and \((12, 18, 27)\) satisfy the conditions.
11. THE GEOMETRY OF THE TRIANGLE

Problem 11.21. Triangle ABC has area 1. Points E, F, G lie, respectively, on sides BC, CA, AB such that AE bisects BF at point R, BF bisects CG at point S, and CG bisects AE at point T. Find the area of the triangle RST. [P2001]

Solution: From the hypothesis, AT = TE, GS = SC, and BR = RF. Denote \( \frac{AF}{AC} = x \), \( \frac{BG}{AB} = y \) and \( \frac{CE}{BC} = z \). From Menelaus's theorem for triangle ABE with crossing line GC we have \( \frac{AG}{GB} \cdot \frac{BC}{CE} \cdot \frac{ET}{TA} = 1 \), and from this \( \frac{AG}{GB} = z \). As a consequence \( y = \frac{GB}{AB} = \frac{1}{z+1} \). In a similar way we obtain the other two equations of the system

\[
x = \frac{1}{y+1}, \quad y = \frac{1}{z+1}, \quad z = \frac{1}{x+1}
\]

Replacing \( z \) from the third equation into the second, and then replacing \( y \) into the first equation we obtain

\[
x = \frac{x+2}{2x+3}
\]

The only positive solution of this equation is

\[
x = -1 + \sqrt{5} \quad \text{2}
\]

which replaced in the other equations gives \( y = z = x \).

Denote by \([ABC]\) the area of the triangle ABC. From \( \frac{[BCS]}{[BCG]} = \frac{1}{2} \) and \( \frac{[BGC]}{[BAC]} = y \) we get \([BCS] = \frac{y}{2} \). Also from \( \frac{[ABR]}{[ABF]} = \frac{1}{2} \) and \( \frac{[ABF]}{[ABC]} = \frac{AF}{AC} = x \) we get \([ABR] = \frac{x}{2} \) and \( \frac{[ATC]}{[AEC]} = \frac{1}{2} \) with \( \frac{[AEC]}{[ABC]} = \frac{CE}{CB} = z \) gives \( [ATC] = \frac{z}{2} \).

We can compute now \([RTS] = [ABC] - [BSC] - [ABR] - [ATC] = 1 - \frac{x+y+z}{2} = 7 - 3\sqrt{5} \).
CHAPTER 12

Geometry

Problem 12.1. Let all plane sections of a certain surface be circles. Show that this surface is a sphere.

Problem 12.2. Consider \( n \) straight lines in the plane. Prove there is an angle between two of these lines which is smaller or equal with \( \frac{2\pi}{n} \).

Solution: If there are two parallel lines the problem is solved. Otherwise consider a point \( A \) and the parallels in \( A \) to all these lines. Like this all the angles between these lines are angles around \( A \). But around \( A \) we have \( n \) angles so at least one is smaller or equal with \( \frac{2\pi}{n} \).

Problem 12.3. If \( ABCD \) is a quadrilateral and \( [ABCD] \) its area, prove that 
\[
[ABCD] \leq \frac{(AB + CD)(BC + DA)}{4}.
\]

Solution: Add the following four inequalities 
\[
AB \cdot BC \geq 2[ABC], \quad CD \cdot DA \geq 2[CDA], \quad AB \cdot DA \geq 2[ABD] \quad \text{and} \quad BC \cdot CD \geq 2[BCD].
\]

Problem 12.4. The octagon \( P_1P_2P_3P_4P_5P_6P_7P_8 \) is inscribed in a circle, with the vertices around the circumference in the given order. Given that the polygon \( P_1P_3P_5P_7 \) is a square of area 5, and the polygon \( P_2P_4P_6P_8 \) is a rectangle of area 4, find the maximum possible area of the octagon. [P2000]

Solution: The side of the square \( P_1P_3P_5P_7 \) is \( \sqrt{5} \), so the radius of the circle is \( \frac{\sqrt{10}}{\sqrt{2}} \). The equations \( P_2P_4 \cdot P_3P_6 = 4 \) and \( P_2P_4^2 + P_4P_6^2 = 10 \) shows the rectangle \( P_2P_4P_6P_8 \) has the sides \( 2\sqrt{2} \) and \( \sqrt{2} \). By symmetry, the area of the octagon can be expressed as 
\[
[P_2P_4P_6P_8] + 2[P_2P_3P_4] + 2[P_4P_5P_6].
\]

Note that \([P_2P_3P_4]\) is \( \sqrt{2} \) times the distance from \( P_3 \) to \( P_2P_4 \), which is maximized when \( P_3 \) lies on the midpoint of arc \( P_2P_4 \); similarly, \([P_4P_5P_6]\) is \( \sqrt{2}/2 \) times the distance from \( P_5 \) to \( P_4P_6 \), which is maximized when \( P_5 \) lies on the midpoint of arc \( P_4P_6 \). Thus the area of the octagon is maximized when \( P_3 \) is the midpoint of arc \( P_2P_4 \) and \( P_5 \) is the midpoint of arc \( P_4P_6 \). In this case, it is easy to calculate that \([P_2P_3P_4] = \sqrt{5} - 1 \) and \([P_4P_5P_6] = \sqrt{5}/2 - 1 \), and so the area of the octagon is \( 3\sqrt{5} \).

Problem 12.5. A \( 2 \times 3 \) rectangle has vertices as \((0,0), (2,0), (0,3), \) and \((2,3)\). It rotates \(90^\circ\) clockwise about the point \((2,0)\). It then rotates \(90^\circ\) clockwise about the point \((5,0)\), then \(90^\circ\) clockwise about the point \((7,0)\), and finally, \(90^\circ\) clockwise
about the point (10,0). (The side originally on the x-axis is now back on the x-axis.) Find the area of the region above the x-axis and below the curve traced out by the point whose initial position is (1,1). [P1991]

Solution: This region is the reunion of

- a triangle with vertices (1,0), (1,1) and (2,0)
- a quarter of a circle of radius $\sqrt{2}$ with center (2,0), starting in (1,1) and going to (3,1)
- a triangle with vertices (5,0), (3,1) and (2,0)
- a quarter of a circle of radius $\sqrt{3}$ with center (5,0), starting in (3,1) and going to (6,2)
- a triangle with vertices (5,0), (6,2) and (7,0)
- a quarter of a circle of radius $\sqrt{3}$ with center (7,0), starting in (6,2) and going to (9,1)
- a triangle with vertices (7,0), (9,1) and (10,0)
- a quarter of a circle of radius $\sqrt{2}$ with center (9,1) and going to (11,1)
- a triangle with vertices (10,0), (11,1) and (11,0)

Therefore the sum $\frac{5\pi}{2} + 6$ of the areas of all these regions is the answer.

Problem 12.6. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube? [P1998]

Solution: Consider the plane containing both the axis of the cone and two opposite vertices of the cube’s bottom face. The cross section of the cone and the cube in this plane consists of a rectangle of sides $s$ and $s\sqrt{2}$ inscribed in an isosceles triangle of base 2 and height 3, where $s$ is the side-length of the cube. (The $s\sqrt{2}$ side of the rectangle lies on the base of the triangle.) Similar triangles yield $\frac{s}{3} = \frac{1 - s\sqrt{2}/2}{1}$, or $s = \frac{9\sqrt{2} - 6}{7}$.

Problem 12.7. Let $s$ be any arc of the unit circle lying entirely in the first quadrant. Let $S_1$ be the area of the region lying below $s$ and above the x-axis and let $S_2$ be the area of the region lying to the right of the y-axis and to the left of $s$. Prove that $S_1 + S_2$ depends only on the arc length, and not on the position, of $s$. [P1998]

Solution: Let $s = CD$ and $A, B, E, F$, be the projections of $C$ and $D$ on the x-axis, respectively y-axis. Let also $\theta$ be the angle $\angle COD$, where $O$ is the origin. Of course $\theta$ and the area of the region bounded by the circle and the segment $CD$ depend only on the length of $s$. It suffices then to show that the sum of areas of the trapezoidal regions $BACD$ and $CDFE$ depends only on $\theta$. If $[BACD]$ is the area of the trapezoidal region $BACD$, and $\alpha = \angle COA$ then we have $[BACD] + [CDFE] = \frac{(AC + BD)AB}{2} + \frac{(CE + DF)EF}{2} =$
\[\frac{(DF + CE)(BD - AC) + (AC + BD)(CE - DF)}{2} = \frac{1}{2}(\cos(\theta + \alpha) + \cos \alpha)(\sin(\theta +\alpha) - \sin \alpha) + \frac{1}{2}(\sin(\theta + \alpha) + \sin \alpha)(\cos \alpha - \cos(\theta + \alpha)) = 2\sin \theta.\]

**Second solution** Denote \(S_3\) the area of the sector COD. Then \(S_1 = S_3 + [COA] - [DOB]\) and \(S_2 = S_3 + [DFO] - [COE]\). But \([COE] = [COA]\) and \([DOF] = [DBO]\), so \(S_1 + S_2 = 2S_3\).

**Problem 12.8.** Right triangle \(ABC\) has right angle at \(C\) and \(\angle BAC = \theta\); the point \(D\) is chosen on \(AB\) so that \(|AC| = |AD| = 1\); the point \(E\) is chosen on \(BC\) so that \(\angle CDE = \theta\). The perpendicular to \(BC\) at \(E\) meets \(AB\) at \(F\). Evaluate \(\lim_{\theta \to 0} |EF|\). \([P1999]\)

**Solution:** In the isosceles triangle \(ACD\), \(\angle ACD = \frac{\pi}{2} - \theta/2\), so \(\angle ECD = \theta/2\) and \(\angle DEC = \pi - \theta - \theta/2\). Also from the isosceles triangle \(ACD\) we have \(CD = 2\sin \theta/2\).

In the triangle \(CDE\) we have the equality
\[
\frac{CD}{\sin(\pi - \frac{3\theta}{2})} = \frac{CE}{\sin \theta},
\]
therefore \(CE = \frac{2\sin \theta \sin \frac{\theta}{2}}{\sin(\pi - \frac{3\theta}{2})}\). The triangles \(BEF\) and \(BCA\) are similar so \(\frac{BE}{BC} = \frac{EF}{AC}\) which implies \(\frac{CE}{BC} = \frac{AC - EF}{AC}\). Thus \(EF = 1 - \frac{CE}{\tan \theta} = \frac{\sin \frac{\theta}{2}}{\sin \frac{3\theta}{2}}\) and the limit is \(1/3\).

**Problem 12.9.** Inscribe a rectangle of base \(b\) and height \(h\) in a circle of radius one, and inscribe an isosceles triangle in the region of the circle cut off by one base of the rectangle (with that side as the base of the triangle). For what value of \(h\) do the rectangle and triangle have the same area? \([P1986]\)

**Solution:** We distinguish two cases.
1. The triangle and the rectangle have only one side in common. The area of the triangle is \(\frac{1 - \frac{h}{2}}{2} \cdot b\) and is equal with the area of the rectangle if \(\frac{1 - h/2}{2} = h\), i.e. \(h = 2/5\).
2. It The intersection of the triangle and the rectangle is a trapezoidal region.

The equality of areas is \(\frac{1 + \frac{h}{2}}{2} = hh\) so \(h = 2/3\).

**Problem 12.10.** Find the least number \(A\) such that for any two squares of combined area 1, a rectangle of area \(A\) exists such that the two squares can be packed in the rectangle (without interior overlap). You may assume that the sides of the squares are parallel to the sides of the rectangle. \([P1996]\)

**Solution:** If \(a \geq b\) are the sides of the squares of combined area 1, then there is \(x \in [0, \frac{\pi}{4}]\) such that \(a = \cos x\) and \(b = \sin x\). The two squares can be packed
without overlapping in a rectangle of sides \( L \geq l \) if \( l \geq a \) and \( L \geq a + b \). Then \( A \) is the maximum of
\[
a(a + b) = \cos x (\cos x + \sin x) = \frac{1}{2} + \frac{\sqrt{2}}{2} \cos \left( \frac{\pi}{4} - 2x \right), \quad \text{i.e.} \quad A = \frac{1}{2} + \frac{\sqrt{2}}{2}.
\]

**Problem 12.11.** Let \( T \) be an acute triangle. Inscribe a rectangle \( R \) in \( T \) with one side along a side of \( T \). Then inscribe a rectangle \( S \) in the triangle formed by the side of \( R \) opposite the side on the boundary of \( T \), and the other two sides of \( T \), with one side along the side of \( R \). For any polygon \( X \), let \( A(X) \) denote the area of \( X \). Find the maximum value, or show that no maximum exists, of \( \frac{A(R) + A(S)}{A(T)} \), where \( T \) ranges over all triangles and \( R, S \) over all rectangles as above. [P1985]

**Solution:** Let \( ABC \) be the triangle and \( DEFG \) the inscribed rectangle with the points \( F \) and \( G \) on the line \( BC \). Denote \( h \) the height of \( T \), \( h_1 \) the height of \( R \) and \( h_2 \) the height of \( S \). We have
\[
\frac{DE}{BC} = \frac{h - h_1}{h} \quad \text{and} \quad \frac{A([ADE])}{A(T)} = \left( \frac{h - h_1}{h} \right)^2.
\]
Also
\[
(0.6) \quad \frac{A(R)}{A(T)} = \frac{DE \cdot h_1}{BC \cdot h/2} = 2 \frac{h_1(h - h_1)}{h^2}.
\]
Similarly,
\[
\frac{A(S)}{A([ADE])} = 2 \frac{h_2(h - h_1 - h_2)}{(h - h_1)^2}, \quad \text{so} \quad \frac{A(S)}{A(T)} = 2 \frac{h_2(h - h_1 - h_2)}{h^2}.
\]

The numbers \( a = \frac{h_1}{h} \) and \( b = \frac{h_2}{h} \) are positive such that \( a + b \leq 1 \) and a simple computation shows that
\[
\frac{A(S) + A(R)}{A(T)} = 2(a + b - a^2 - ab - b^2).
\]
Let \( S = a + b \) and \( P = ab \). Then
\[
\frac{A(S) + A(R)}{A(T)} = 2(S - S^2 + P) \leq 2S - 2S^2 + \frac{S^2}{2} = 2S - \frac{3S^2}{2}.
\]
But this last expression has the maximum \( 2/3 \) attained for \( S = 2/3 \), and in the previous inequality, the equality is obtained for \( P = S^2/4 = 1/9 \). So
\[
\frac{A(R) + A(S)}{A(T)}
\]
has the maximum \( 2/3 \) attained in the case \( \frac{h_1}{h} = \frac{1}{3} \) and \( \frac{h_1}{h} = \frac{1}{3} \).

**Problem 12.12.** A rectangle, \( HOMF \), has sides \( HO = 11 \) and \( OM = 5 \). A triangle \( ABC \) has \( H \) as the intersection of the altitudes, \( O \) the center of the circumscribed circle, \( M \) the midpoint of \( BC \), and \( F \) the foot of the altitude from \( A \). What is the length of \( BC \)? [P1997]

**Solution:** Denote by \( a \) the length of \( BC \). From \( AO = OC \), and using Pithagora’s theorem in the triangles \( AHO \) and \( MOC \) we get
\[
AH^2 + 11^2 = 5^2 + \frac{a^2}{4}, \quad \text{so}
\]
\[
(0.7) \quad AH^2 = \frac{a^2}{4} - 96.
\]
In the triangle \( AFC \), \( AF = \left( \frac{a}{2} + 11 \right) \tan C \) and in the triangle \( BHF \) after the remark \( \angle BHF = \angle C \), \( BF = 5 \tan C \). So
\[
AF = \left( \frac{a}{2} + 11 \right) BF = 5 \left( \frac{a^2}{4} - 121 \right).
\]
But \( AH = AF - 5 \), thus
\[
(0.8) \quad AH = 5 \left( \frac{a^2}{4} - 121 \right) - 5
\]
With the notation \( x = AH = \frac{1}{5} \left( \frac{a^2}{4} - 121 \right) - 5 \), from 0.7 and 0.8 we obtain the equation \( x^2 = 5x + 50 \) with the only positive solution \( x = 10 \) and consequently \( a = 28 \).

**Problem 12.13.** Let \( \mathcal{F} \) be a finite collection of open discs in \( \mathbb{R}^2 \) whose union contains a set \( E \subseteq \mathbb{R}^2 \). Prove there is a pairwise disjoint subcollection \( D_1, \ldots, D_n \) in \( \mathcal{F} \) such that \( E \subseteq \bigcup_{j=1}^{n} 3D_j \). Here, if \( D \) is the disc of radius \( r \) and center \( P \), then \( 3D \) is the disc of radius \( 3r \) and center \( P \). [P1998]

**Solution:** Let \( D_1 \) be the disc of \( \mathcal{F} \) with the greatest radius \( r \). We remark that \( 3D_1 \) contains all the discs which intersect \( D_1 \). Indeed if a disc \( D \) with radius \( s \) intersects \( D_1 \), then any point in \( D \) is at a distance smaller than \( r + 2s \leq 3r \) from the center of \( D_1 \). We take out from \( \mathcal{F} \) the disc \( D_1 \) together with all the discs which intersect him from \( \mathcal{F} \), and denote by \( \mathcal{F}_1 \) this set. Obviously \( \mathcal{F}_1 \) has less elements than \( \mathcal{F} \). Let \( D_2 \) be the disc in \( \mathcal{F}_1 \) with the greatest radius. Then all the discs which intersect \( D_1 \) are inside \( 3D_1 \). Taking out \( D_1 \) with all the discs which intersect him from \( \mathcal{F}_1 \) we get a set \( \mathcal{F}_2 \). Repeating this algorithm we obtain the sequence \( D_1, D_2, \ldots, D_n \).

**Problem 12.14.** Let \( d_1, d_2 \) be an angle (less than \( 90^\circ \)) and \( k > 0 \) a fixed real number. Find the set of points \( A \) inside the angle with the property

- a) \( AM - AN = k \), where \( M \) is the foot of the perpendicular from \( A \) to \( d_1 \), and \( N \) is the foot of the perpendicular from \( A \) to \( d_2 \).
- b) \( AM + AN = k \)

**Problem 12.15.** a) Let \( ABC \) be a triangle and \( M \) a point on \( BC \) such that \( BM = CM \). Prove that \( AM^2 = \frac{AB^2 + AC^2 - BC^2}{4} \).

b) Prove that the quadrilateral \( ABCD \) is a parallelogram if and only if the sum of the squares of the sides is equal with the sum of squares of the diagonals.

**Problem 12.16.** Let \( ABC \) be a triangle with a right angle in \( A \) and \( AB = \frac{AC}{2} \). Prove that we can cut the triangle \( ABC \) in five identical triangles.

**Solution:** Let \( E \) be the midpoint of \( AC \), \( D \) the foot of the perpendicular from \( A \) to \( BC \), \( F \) the foot of the perpendicular from \( E \) to \( BC \), and \( G \) the intersection of the parallel from \( F \) to \( AC \) with \( AD \). The 5 triangles \( ABD, AGE, GDF, GEF \) and \( EFC \) are identical.

**Problem 12.17.** Let \( ABC \) be a triangle. Prove that the measure of the angle \( A \) is less than \( 90^\circ \) if and only if there is a point \( M \) such that \( MA^2 = MB^2 + MC^2 \).

**Problem 12.18.** Show that in a convex pentagon the ratio between the greatest diagonal and the smallest side is greater or equal than \( \frac{1 + \sqrt{5}}{2} \).

**Problem 12.19.** Let \( ABC \) be a triangle and the function \( f : \text{Int}(ABC) \rightarrow \mathbb{R} \), \( f(X) = 3AX^2 + BX^2 + 2CX^2 \). Find the minimum of this function and the point where this minimum is realized.

**Problem 12.20.** The sum of the areas of some squares is 1. Prove that this squares can be arranged without superpositions inside a square of area 2.
Problem 12.21. Consider in the plane a finite number of polygons, such that any two of them have at least a common point. Prove there is a straight line which intersects all the polygons.

Problem 12.22. On the surface of a cube of side 1 there is a line (frântă) such that on each face there is at least one segment. Prove that the length of the line is at least $3\sqrt{2}$.

Problem 12.23. Prove that a triangle is equilateral if and only if the sum of the distances from a point in the interior of the triangle to the sides is constant.

Problem 12.24. There are four straight lines in the plane. No two are parallel and no three meet at one point. Along each line a pedestrian walks at a constant speed. It is known that the first pedestrian meets the second, the third and the fourth ones, and the second pedestrian meets the third and the fourth ones. Prove that the third pedestrian meets the fourth one.

Problem 12.25. Let $ABCD$ be a quadrilateral and $E, F$ the midpoints of the diagonals. Prove that

$$4EF^2 = AB^2 + BC^2 + CD^2 + DA^2 - AC^2 - BD^2.$$ 

Consequence: In any quadrilateral the sum of the squares of the sides is greater or equal with the sum of squares of the diagonals.

Solution: Use the problem 10.13.

Problem 12.26. Let $A, B$ be two fixed points and $d$ a fixed straight line. Find a point $M$ on $d$ such that $AM^2 + BM^2$ is minimum.

Solution: Let $C$ be the midpoint of $AB$. Then $MC^2 = \frac{AM^2 + BM^2}{2} - \frac{AB^2}{4}$. Therefore $AM^2 + BM^2$ is minimum when $CM$ is minimum. And this happens for $M$ leg of the perpendicular from $C$ to $d$.

Problem 12.27. Let $ABC$ be a triangle, and consider the equilateral triangles $MAB, NAC$ and $PBC$, with $M$ and $C$ in opposite half-planes with respect to $AB$, etc. Prove that one can make an equilateral triangle with the segments $AP, BN$ and $CM$.

Solution: Since $AC = NC$, the angles $ACP$ and $NCB$ are equal, and $CP = BC$, the triangles $ACP$ and $NCB$ are congruent, so $AP = BN$. In a similar way, considering the triangles $MBC$ and $ABP$, we prove $MC = AP$.

Problem 12.28. Let $A_k, k = \overline{1,n}$ be $n$ given points. Find a point $M$ such that the sum $\sum_{k=1}^{n} MA_k^2$ is minimum.

Solution: Consider the coordinates $A_k(a_k, b_k), M(x, y)$. The sum to minimize is

$$\sum_{k=1}^{n} [(x-a_k)^2 + (y-b_k)^2] = nx^2 - 2x \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k^2 + ny^2 - 2y \sum_{k=1}^{n} b_k + \sum_{k=1}^{n} b_k^2.$$ 

This function of $(x, y)$ is minimum for $x = \frac{1}{n} \sum_{k=1}^{n} a_k$, $y = \frac{1}{n} \sum_{k=1}^{n} b_k$.

Problem 12.29. Prove that the sums of squares of the lengths of the sides of a convex polygon circumscribed to a circle is smaller or equal than $9R^2$. 

Problem 12.30. Let $ABC$ be a triangle and $M$, respectively $N, P$, be the mid-point of the segment $BC$, respectively $AC, AB$. Prove that $AM$ is perpendicular to $BN$ if and only if $AM^2 + BN^2 = CP^2$.

Problem 12.31. Let $ABC$ be a triangle and $O$ a point in the interior of the triangle. Let $M$, respectively $N, P$, be the leg of the perpendicular from $O$ to $AB$, respectively, $AC, BC$. Prove that the sum $OM + ON + OP$ is independent of the position of $O$ if and only if the triangle $ABC$ is equilateral.

Problem 12.32. Let $A, B$ be two points, and $k$ a positive constant. Determine the locus of the point $M$ with the property $MA^2 + MB^2 = k$. 

CHAPTER 13

Trigonometry

PROBLEM 13.1. Prove that \( \sin 10^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{8} \).

Solution : We have
\[
8 \sin 10 \sin 50 \sin 70 = 4(\cos 40 - \cos 60) \sin 70 = 4 \sin 70 \cos 40 - 2 \sin 70 \\
= 2(\sin 110 + \sin 30) - 2 \sin 70 = 2(\sin 110 - \sin 70) + 1 = 1
\]

PROBLEM 13.2. Find the exact value of \( \cos \frac{\pi}{5} \).

Solution : We observe that \( \cos \frac{3\pi}{5} = -\cos \frac{2\pi}{5} \). Denoting \( t = \cos \frac{\pi}{5} \) and using the triple and double angle formulae we obtain \( 4t^3 - 3t = 1 - 2t^2 \). We factor the equation as \( (t + 1)(4t^2 - 2t - 1) = 0 \) and we obtain \( t = \frac{1 + \sqrt{5}}{2} \).

Second solution: We have \( \cos \frac{2\pi}{5} - \cos \frac{\pi}{5} = \cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -2 \cos \frac{\pi}{5} \cos \frac{2\pi}{5} \). From here \( t = (2t^2 - 1)(2t + 1) \), etc.

Third solution: We have \( \cos \frac{\pi}{5} = -\cos \frac{4\pi}{5} = 1 - 2 \cos^2 \frac{2\pi}{5} = -8 \cos^4 \frac{\pi}{5} + 8 \cos^2 \frac{\pi}{5} - 1 \)

Therefore \( \cos \frac{\pi}{5} \) is a zero of the polynomial \( P(t) = 8t^4 - 8t^2 + t + 1 = (t + 1)(2t - 1)(4t^2 - 2t - 1) \) and consequently it is \( \frac{1 + \sqrt{5}}{4} \).

Fourth solution: Set \( z = \cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \) and we have \( z^5 = -1 \). It follows that \( z^4 - z^3 + z^2 - z = 1 = 0 \), equation which we write
\[
\left( z + \frac{1}{z} \right)^2 - \left( z + \frac{1}{z} \right) - 1 = 0
\]
From here \( z + \frac{1}{z} = \frac{1 \pm \sqrt{5}}{2} \) and then
\[
z = \frac{w \pm i\sqrt{4 - w^2}}{2}.
\]
Finally \( \cos \frac{\pi}{5} = \Re z = \frac{w}{2} = \frac{1 + \sqrt{5}}{4} \).

Fifth solution. Use the Chebyshev polynomial \( T_5(x) = 16x^5 - 20x^3 + 5x + 1 = (4x^2 - 2x - 1)(4x^3 + 2x^3 - 3x - 1) \).

PROBLEM 13.3. Let \( a, b, c \) be real numbers such that
\[
\sin a + \sin b + \sin c = \cos a + \cos b + \cos c = 0
\]

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Prove that $\tan 3a = \tan 3b = \tan 3c$.

**Solution:** We have $1 = \sin^2 a + \cos^2 a = (-\sin a - \sin b)^2 + (-\cos a - \cos b)^2 = 2 + 2\cos(a-b)$, so $\cos(a-b) = -\frac{1}{2}$. It follows there is $k \in \mathbb{Z}$ such that $a-b = 2k\pi \pm \frac{2\pi}{3}$. Then $\tan 3a = \tan(3b+6k\pi \pm \frac{2\pi}{3}) = \tan(3b)$.

**Observation.** The points in the complex plane $z_1 = \cos a + i \sin a$, $z_2 = \cos b + i \sin b$, and $z_3 = \cos c + i \sin c$ are the vertices of an equilateral triangle inscribed in the unit circle.

**Problem 13.4.** Find the positive integers $x, y, z$ such that $\arctan x + \arctan y + \arctan z = \frac{5\pi}{4}$.

**Solution:** For any positive integer $u$ we have $\frac{\pi}{4} \leq \arctan u < \frac{\pi}{2}$, hence $S = \arctan x + \arctan y + \arctan z \in \left[ \frac{3\pi}{4}, \frac{3\pi}{2} \right)$. On this interval the function $\tan$ is injective, so the equation is equivalent with

$$\tan S = 1 \iff \frac{x+y+z-xyz}{1-xy-yz-zx} = 1$$

Without loss of generality we can assume $x \leq y \leq z$. Solving the last equation for $x$ we obtain

$$x = 1 + \frac{2y+2z}{yz-y-z+1} > 1$$

We examine all the possible cases

- $x = 2 \iff (y-3)(z-3) = 10$ with the solutions $y = 4, z = 13$ and $y = 5, z = 8$.
- $x = 3 \iff (y-2)(z-2) = 5$ with the solution $y = 3, z = 7$.
- $x \geq 4$ In this case we obtain $\frac{2y+2z}{yz-y-z+1} \geq 3 \iff (y-2)(z-2) \leq \frac{y+z}{3}$. From here it follows that necessarily $y = z = 4$, but for these values $x$ is not integer.

Therefore the solutions are all the permutations of the triples $(2, 4, 13)$, $(2, 5, 8)$ and $(3, 3, 7)$.

**Problem 13.5.** Consider the family of functions of real variable

$$f_m(x) = \sqrt{\cos^4 x + m \sin^2 x} + \sqrt{\sin^4 x + m \cos^2 x}$$

Find the values of $m$ for which $f_m$ is a constant function.

**Solution:** The equation $f_m(0) = f_m \left( \frac{\pi}{4} \right)$ has the solutions $m = 0$ and $m = 4$. A simple computation shows that $f_0(x) = 1$ and $f_4(x) = 3$.

**Problem 13.6.** For any $b, c$ real numbers, prove that $\cos(b+c) + \sqrt{2}(\sin b + \sin c) \leq 2$.

**Solution:** The inequality can be arranged $(\sqrt{2} - \sin c)\sin b + \cos c \cos b \leq 2 - \sin b$. We use the inequality $A \cos b + B \sin b \leq \sqrt{A^2 + B^2}$ and it suffices to prove $\sqrt{(\sqrt{2} - \sin c)^2 + \cos^2 c} \leq 2 - \sqrt{2} \sin c$ which is equivalent with $(\sqrt{2} \sin c - 1)^2 \geq 0$.
Problem 13.7. Let \( x, y, z \) be real numbers different from \( \pm \frac{1}{\sqrt{3}} \), such that \( x + y + z = xyz \). Prove that
\[
\frac{3x - x^3}{1 - 3x^2} + \frac{3y - y^3}{1 - 3y^2} + \frac{3z - z^3}{1 - 3z^2} = \frac{3x - x^3}{1 - 3x^2} \frac{3y - y^3}{1 - 3y^2} \frac{3z - z^3}{1 - 3z^2}
\]

Solution: If \( x = \tan a, y = \tan b, z = \tan c \), then
\[
\tan(a + b + c) = \frac{\tan a + \tan b + \tan c - \tan a \tan b \tan c}{1 - \tan a \tan b - \tan b \tan c - \tan c \tan a} = 0,
\]
so \( a + b + c \) is a multiple of \( \pi \). But \( 3(a + b + c) \) is also a multiple of \( \pi \), hence \( \tan(3a + 3b + 3c) = 0 \) and consequently \( \tan 3a + \tan 3b + \tan 3c = \tan 3a \tan 3b \tan 3c \).

We end the proof with the remark \( \tan 3a = \frac{3 \tan a - \tan^3 a}{1 - 3 \tan^2 a} \).

Problem 13.8. If \( x \) and \( y \) are rational numbers such that \( \tan \pi x = y \), prove that \( x = \frac{k}{4} \) for some integer \( k \) not congruent to 2 (mod 4).

Problem 13.9. Let \( n \geq 2 \) be an integer. Evaluate \( \sum_{k=0}^{n-2} 2^k \tan \frac{\pi}{2n-2k} \).

Solution: Using \( \tan x = \cot x - 2 \cot 2x \) we have
\[
\sum_{k=0}^{n-2} 2^k \tan \frac{\pi}{2n-2k} = \sum_{k=0}^{n-2} \left( 2^k \cot \frac{\pi}{2n-2k} - 2^{k+1} \cot \frac{\pi}{2n-(k+1)} \right) = \cot \frac{\pi}{2n} - 2^{n-1} \cot \frac{\pi}{2} = \cot \frac{\pi}{2n}.
\]

Problem 13.10. If \( k \) is a positive integer and \( \theta = \frac{2\pi j}{k} \), for \( j \in \{1, 2, \ldots, k-1\} \), show that
\[
\sum_{n=1}^{[(k-1)/2]} \cos n\theta = -2 + (-1)^{k+j} + (-1)^j / 4
\]

Solution: For any integer \( p \) the following indentity holds
\[
\cos \theta + \cos 2\theta + \ldots + \cos p\theta = \frac{\sin \left(\frac{2p+1}{2}\theta\right) - \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}}
\]
If \( k \) is odd, say \( k = 2p + 1 \), then
\[
\sum_{n=1}^{p} \cos n\theta = \frac{\sin \pi j - \sin \frac{\theta}{2}}{2 \sin \frac{\theta}{2}} = -\frac{1}{2}
\]
For \( k \) even, say \( k = 2p \),
\[
\sum_{n=1}^{p-1} \cos n\theta = \frac{\sin(\pi j - \frac{\theta}{2}) - 1}{2 \sin \frac{\theta}{2}} = -(-1)^j - \frac{1}{2}
\]
Problem 13.11. Show that \((\sin x)^2 \leq \sin(x^2)\) for \(0 \leq x \leq \frac{\sqrt{\pi}}{2}\).

Solution: The inequality is equivalent with \(f(x) = x^2 - \arcsin(\sin^2 x) \geq 0\). We have \(f'(x) = 2x - \frac{2\sin x \cos x}{\sqrt{1 - \sin^4 x}} = 2x - \frac{2\sin x}{\sqrt{1 + \sin^2 x}} \geq 2 - 2\sin x \geq 0\) for all \(x \in \left[0, \frac{\sqrt{\pi}}{2}\right]\). Then \(f(x) \geq f(0) = 0\) for all \(x \in \left[0, \frac{\sqrt{\pi}}{2}\right]\).

Problem 13.12. Solve the equation \(\sqrt{\sin x} + \sqrt{\cos x} = t\).

Solution: Obviously, \(t\) must be positive and without any restriction of the generality one can suppose \(x \in \left[0, \frac{\pi}{2}\right]\). The other solutions will be obtained using the fact that the function \(f(x) = \sqrt{\sin x} + \sqrt{\cos x}\) has the period \(2\pi\). Denote \(u = \sin x + \cos x\). Taking its square, the equation to solve is equivalent with \(u + 2\sqrt{\frac{u^2 - 1}{2}} = t^2\). One can arrange this equation under the equivalent form \(u^2 + 2ut^2 - t^4 - 2 = 0\). Since \(u\) is positive the only convenient solution is \(u = \sqrt{2(t^4 + 1)} - t^2\). This can be put under the form \(\sin\left(x + \frac{\pi}{4}\right) = \sqrt{t^4 + 1 - \frac{t^4}{\sqrt{2}}}\).

For \(x \in \left[0, \frac{\pi}{2}\right]\), we have \(\sin\left(x + \frac{\pi}{4}\right) \in \left[\frac{\sqrt{2}}{2}, 1\right]\). Therefore \(t\) is subject to the condition \(\frac{\sqrt{2}}{2} \leq \sqrt{t^4 + 1} - \frac{t^2}{\sqrt{2}} \leq 1\). The left inequality is always satisfied, and the right one yields \(t \in [0, 2^{1/4}]\). For such values of \(t\), the solutions are \(x = \arcsin\left(\sqrt{t^4 + 1 - \frac{t^4}{\sqrt{2}}} - \frac{\pi}{2} + 2k\pi\right)\), for all integers \(k\).

Problem 13.13. If \(A\) and \(B\) are non-negative numbers, then

\[\arctan A - \arctan B = \arctan \frac{A - B}{1 + AB}\].

Solution: The numbers \(A\) and \(B\) are both in the interval \([0, \pi/2]\), so \(\arctan A - \arctan B \in (-\pi/2, \pi/2)\). For \(A, B \geq 0\) we have \(1 + AB \neq 0\), so \(\arctan \frac{A - B}{1 + AB}\) is well defined and also lies in the interval \((-\pi/2, \pi/2)\). The equality to prove is equivalent then with the trivial identity \(\tan(\arctan A - \arctan B) = \frac{A - B}{1 + AB}\).

Problem 13.14. For any \(x \in \mathbb{R}\), \(|\sin x| \leq |x|\).

Solution: Consequence of the fact that \(\sin x \leq x\), for all \(x \geq 0\).

Problem 13.15. Prove that \(\cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \frac{1}{2}\).

Solution: We have \(\cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \cos \frac{\pi}{7} + \cos \frac{3\pi}{7} + \cos \frac{5\pi}{7} = \frac{1}{2}\sin \frac{\pi}{7} (\sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} - \sin \frac{2\pi}{7} + \sin \frac{6\pi}{7} - \sin \frac{4\pi}{7}) = \frac{1}{2} \sin \frac{\pi}{7} = \frac{1}{2}\).
Problem 13.16. Prove that \( \tan \frac{3 \pi}{11} + 4 \sin \frac{2 \pi}{11} = \sqrt{11} \).

Solution : We transform this equation in the following equivalent forms:

\[
\tan^2 \frac{3 \pi}{11} + 8 \tan \frac{3 \pi}{11} \sin \frac{2 \pi}{11} + 16 \sin^2 \frac{2 \pi}{11} = 11
\]

\[
\cos^2 \frac{3 \pi}{11} + 8 \cos \frac{3 \pi}{11} + 16 \sin^2 \frac{2 \pi}{11} = 12
\]

\[
1 + 8 \sin \frac{2 \pi}{11} \sin \frac{3 \pi}{11} + 16 \sin^2 \frac{2 \pi}{11} \cos \frac{3 \pi}{11} = 12 \cos \frac{3 \pi}{11}
\]

\[
1 + 4 \sin \frac{2 \pi}{11} \sin \frac{6 \pi}{11} + 4 \left(1 - \cos \frac{4 \pi}{11}\right) \left(1 + \cos \frac{6 \pi}{11}\right) = 6 \left(1 + \cos \frac{6 \pi}{11}\right)
\]

\[
-2 \cos \frac{2 \pi}{11} - 2 \cos \frac{4 \pi}{11} - 2 \cos \frac{6 \pi}{11} - 2 \cos \frac{8 \pi}{11} - 2 \cos \frac{10 \pi}{11} = 1
\]

Multiplying by \( \sin \frac{\pi}{11} \) and transforming the products in differences the last equality can be written as

\[
\sin \frac{\pi}{11} - \sin \frac{3 \pi}{11} + \sin \frac{3 \pi}{11} - \sin \frac{5 \pi}{11} + \sin \frac{5 \pi}{11} - \sin \frac{7 \pi}{11} + \sin \frac{7 \pi}{11} - \sin \frac{9 \pi}{11} + \sin \frac{9 \pi}{11} - \sin \frac{11 \pi}{11} = \sin \frac{\pi}{11}
\]

Problem 13.17. Solve the equation \( \sin 3x \sin^3 x + \cos 3x \cos^3 x = \frac{1}{8} \).

Solution : It can be shown that the left hand side is \( \cos^3 2x \).

Problem 13.18. Find the positive integers \( n \) such that

\[
\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} = \sqrt{\frac{n}{2}}
\]

Solution : If \( n \) is a solution, then taking the square of the equation we obtain

\[
\sin \frac{\pi}{n} = \frac{n - 4}{4}
\]

Necessarily, \( n \leq 8 \) and examining the cases we see that \( n = 6 \) is the only solution.

Problem 13.19. Prove that

a) \( \cos \frac{\pi}{7} + \cos \frac{3 \pi}{7} + \cos \frac{5 \pi}{7} = \frac{1}{2} \)

b) \( \cos^2 \frac{\pi}{7} + \cos^2 \frac{3 \pi}{7} + \cos^2 \frac{5 \pi}{7} = \frac{5}{4} \)

c) \( \cos^3 \frac{\pi}{7} + \cos^3 \frac{3 \pi}{7} + \cos^3 \frac{5 \pi}{7} = \frac{1}{4} \)

d) \( \cos^n \frac{\pi}{7} + \cos^n \frac{3 \pi}{7} + \cos^n \frac{5 \pi}{7} \) is a rational number for any \( n \) positive integer
CHAPTER 14

Probability

Problem 14.1. We pick one number from the set \( \{1, 2, 3, ..., 2003\} \). What is the probability that this number is a multiple of 2, or 3, or 5?

Problem 14.2. We cut a segment of length 1 in 3 pieces. What is the probability that we can make a triangle with this 3 segments?

Problem 14.3. You have coins \( C_1, C_2, \ldots, C_n \). For each \( k \), \( C_k \) is biased so that, when tossed, it has probability \( \frac{1}{2k+1} \) of falling heads. If the \( n \) coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of \( n \). [P2001]

Solution: The answer is \( \frac{n}{2n+1} \) and we will prove it by induction. For \( n = 1 \), the probability that the number of heads is odd coincides with the probability of falling head, so it is \( \frac{1}{3} \). Suppose that if \( k \) coins are tossed, the probability that the number of heads is odd is \( \frac{n}{2n+1} \). Then if \( k + 1 \) heads are tossed, the event of having an odd number of heads is the reunion of two disjoint events, the number of heads between the first \( k \) coins is odd and \( C_{k+1} \) is not a head and the number of heads between the first \( k \) coins is even and the \( C_{k+1} \) is a head. Hence the probability is

\[
\frac{k}{2k+1} \left(1 - \frac{1}{2k+3}\right) + \frac{1}{2k+3} = \frac{k+1}{2k+3}
\]

Problem 14.4. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form \( \frac{a\sqrt{b} + c}{d} \), where \( a, b, c, d \) are integers. [P1989]

Solution: Without any restriction of the generality we can suppose that the side of the square has the length 4. If \( R \) is the region consisting of the which are nearer to the center thn to any edge, the required probability is \( p = \frac{\text{Area}R}{\text{Areaofthesquare}} \).

Using the symmetry, \( p = \frac{\text{Area}R \cap [CAB]}{\text{Area}[CAB]} \), where \([CAB]\) is the triangle determined by the center \( C \) of the square, the vertex \( B \) of the square and the middle \( A \) of the side \( BD \) of the square. Let us consider a system of coordinates with the origin \( O \) in the middle of \( CA \), the \( x \)-axis parallel to \( AB \) and \( CA \) as the \( y \)-axis. Then \( R \cap [CAB] \) is the region bounded by the parabola \( y = x^2 \), the \( y \)-axis and the
line \( y = 1 - x \). So, if \( x_1 = \frac{\sqrt{5} - 1}{2} \) is the positive solution of \( 1 - x = x^2 \), then

\[
\text{Area}\, R \cap \{CAB\} = \int_0^{x_1} (1 - x - x^2)\,dx = \frac{5\sqrt{5} - 7}{12} \\
\text{and } p = \frac{5\sqrt{5} - 7}{24}.
\]

**Problem 14.5.** A micobe either splits into two perfect copies of itself or else die. If the probability of splitting is \( p \), what is the probability that one micobe will produce an everlasting colony?

**Problem 14.6.** Device an experiment which uses only tosses of a fair coin, but which has success probability \( 1/3 \). Do the same for any success probability \( p \in [0, 1] \).

**Problem 14.7.** Find the probability when a bridge game is dealt that each player has exactly one ace.

\[
\text{Solution} : \quad 4! \frac{12!12!12!12!}{52!} \frac{48!}{13!13!13!13!}
\]

**Problem 14.8.** What is the probability that a poker hand will have

a) one pair?

b) no pair (five different face values not all of the same suit).

c) two pairs

d) three of a kind

e) straight (five cards in sequence, but not all of the same suit).

f) flush (five cards of the same suit but not in sequence).

g) full house (three cards of one face value, and two cards of another face value).

h) four of a kind.

i) straight flush.

\[
\text{Solution} : \quad a) \quad \frac{C_4^1 \cdot 4 \cdot C_4^2 \cdot 4^3}{C_5^5}
\]

**Problem 14.9.** How many diagonals does a polygon of \( n \) sides have?

\[
\text{Solution} : \quad \frac{n(n - 3)}{2} = C_n^2 - n
\]

**Problem 14.10.** The 11 digits 1, 2, 2, 2, 3, 3, 3, 4, 4 are permuted in all distinguishable ways. How many permutations begin with 22?

\[
\text{Solution} : \quad \frac{9!}{3!4!1!1!}
\]

**Problem 14.11.** Each permutation of the digits 1, 2, 3, 4, 5, 6 determines a six-digit number. If the numbers corresponding to all possible permutations are listed in order of increasing magnitude, which is the 417th?

\[
\text{Solution} : \quad \text{The first 120=5! of these numbers start with 1, the next 120 begin with 2, those between the ranks 241 and 360 begin with a 3, so at the position 417 we have a number starting with a 4. We continue on this idea and find the number 432516.}
\]

Problem 14.12. The six digits 1,1,1,2,3,3 are permuted, and we list the corresponding six-digits number in order of increasing magnitude. How far down in the list is the number 321311?

Problem 14.13. We have two each of \(n\) different objects, \(2n\) objects altogether. How many distinguishable selections of four objects are there?

Solution: We have \(C_n^4\) selections of four different objects, \(3C_n^3\) selections with 2 alike and 2 different objects, \(C_n^2\) selections of 2 pairs of two alike objects. The number of selections is the total.

Problem 14.14. The 11 letters of the word Mississippi are scrambled and then arranged in some order.

a) What is the probability that the four i’s are consecutive letters in the resulting arrangement?

b) What is the probability that the 4 i’s are consecutive and the 4 s’s are consecutive?

c) What is the probability that no three consecutive letters are alike?

Problem 14.15. A poker player holds a pair of aces and a king, queen, and jack. He discards three cards, holding his pair, and draws three more cards from the deck of 47 cards. What is the probability that his hand contains

a) three aces after the draw?

b) two pairs, ace high, after the draw?

Problem 14.16. In a bridge game, what is the probability

a) that you and your partner together have exactly \(k\) aces, where \(k = 0, 1, 2, 3, 4\)?

b) of a \(4-3-3-3\), \(5-4-3-1\), \(4-4-3-2\) distribution?

Problem 14.17. You and your partner in bridge are declarers and hold nine spades, including the ace and the king. The defenders hold four spades, inlcuding the queen. What is the probability that the distribution of the four spades in the opposing hands is

(a) four in one hand, none in the other?

(b) three in one hand, one in the other?

c) two in one hand, two in the other?

d) You know that the queen will fall when you lead the ace and king if the four spades are divided equally between the opposing hands or if the queen is the only spade in one of them. Find the probability of the queen falling on the lead of the ace and the king.

Problem 14.18. Let \(A\) be a set with \(n\) elements and \(B\) a set with \(m\) elements.

a) Find the number of functions \(f: A \rightarrow B\)

b) Find the number of injective functions \(f: A \rightarrow B\)

c) Find the number of surjective functions \(f: A \rightarrow B\)

Problem 14.19. Three balls are dropped into three boxes. Find the probability that exactly one box will be empty.

Solution: Let \(b_1, b_2, b_3\) be the three balls and \(B_1, B_2, B_3\) the three boxes. We have to find the number of functions \(f: \{b_1, b_2, b_3\} \rightarrow \{B_1, B_2, B_3\}\) with the image a set with two elements. And this is 3 times the number of surjective functions from a set with 3 elements to one with 2 elements.
14. Problem 14.20. Find the number of 13-card bridge hands that will contain the A K Q J 10 of at least one suit.

**Solution:** Let $E_S, E_C, E_H$, respectively $E_D$, be the event that the A K Q J 10 of spades, clubs, hearts, respectively diamonds, are in the 13-card hand. Denote by $|E|$ the number of elements of $E$. Then $|E_S \cup E_H \cup E_D \cup E_C| = \sum |E_S| - \sum |E_S \cap H| + \sum |E_S \cap H \cap D| - |E_S \cap H \cap D \cap C|$. But $|E_S| = \binom{8}{3} \cdot \binom{47}{4}$, $|E_S \cap H| = \binom{3}{3} \cdot \binom{42}{4}$, $|E_S \cap H \cap D| = 0$, hence $|E_S \cup H \cup D \cup C| = 4 \binom{8}{3} \cdot \binom{47}{4} - 6 \binom{3}{3} \cdot \binom{42}{4}$.

**Problem 14.21.** If a three digit number is chosen, find the probability that exactly 1 digit will be greater than 5.

**Problem 14.22.** A urn contains 10 balls numbered from 1 to 10. Five balls are drawn without replacement. Find the probability that the second largest of the five numbers will be 8.

**Problem 14.23.** Consider a set of $n$ elements. The number of
a) ordered samples of size $r$, with replacement, is $n^r$

b) ordered samples of size $r$, without replacement, is $\frac{n!}{(n-r)!}$

c) unordered samples of size $r$, without replacement, is $\binom{n}{r}$

d) unordered samples of size $r$, with replacement, is $\binom{n+r-1}{r}$

**Problem 14.24.** An experiment consist of drawing 10 cards from an ordinary 52-card pack.

a) If the drawing is done without replacement, find the probability that no two cards will have the same face value.

b) If the drawing is done without replacement, find the probability that at least 9 cards will be of the same suit.

**Problem 14.25.** In the $m+w$ seats of a row are seated $m$ men and $w$ women. Find the probability that all the women will be adjacent.

**Problem 14.26.** Eight cards are drawn without replacement from an ordinary deck. Find the probability of obtaining exactly three aces or exactly three kings.

**Problem 14.27.** An urn contains $n$ tickets numbered 1, 2, 3, ..., $n$. The tickets are shuffled thoroughly and then drawn one by one without replacement. If the ticket numbered $r$ appears in the $r$-th drawing, this is denoted as a match. Show that the probability of at least one match is

$$1 - \frac{1}{2!} + \frac{1}{3!} - ... + (-1)^{n-1} \frac{1}{n!}$$

**Problem 14.28.** A “language” consists of three “words”, $W_1 = a$, $W_2 = ba$, $W_3 = bb$. Let $N(k)$ be the number of “sentences” using exactly $k$ letters, e.g. $N(1) = 1$, (i.e a), $N(2) = 3$ (aa, ba, bb), $N(3) = 5$, (aaa, aba, abb, baa, bba); no space is allowed between words. Find a formula for $N(k)$.

**Solution:** Prove that $N(k) = N(k-1) + 2N(k-2)$.

**Problem 14.29.** Assume that a person’s birthday is equally likely to fall on any of the 365 days in a year (neglect leap years). If $r$ people are selected, find the probability that all $r$ birthdays will be different. It turns out that the probability is
less than 1/2 for \( r \geq 23 \), so that in a class of 23 or more students the odds are that two or more people will have the same birthday.

**Problem 14.30.** Let \( \Omega \) be a set of \( n \) elements. How many ways are there of selecting ordered pairs \((A, B)\) of subsets of \( \Omega \) such that \( A \subset B \)?

**Problem 14.31.** Let \( \Omega \) be a finite set with \( n \) elements. A partition of \( \Omega \) is an (unordered) set \( \{A_1, A_2, ..., A_k\} \), where \( A_i \) are nonempty sets whose union is \( \Omega \). Let \( g(n) \) be the number of partitions of a set with \( n \) elements.

\( \)  
(a) Show that \( g(n) = \sum_{k=0}^{n-1} C_{n-1}^k g(k) \) (define \( g(0) = 1 \)).

(b) Show that \( g(n) = e^{-1} \sum_{k=0}^{\infty} \frac{k^n}{n!} \).

**Problem 14.32.** Find the probability that the sum of \( n \) randomly chosen integers is odd.
CHAPTER 15

Functional equations and inequations

1. The Cauchy equation

Problem 15.1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( f(x + y) = f(x) + f(y) \) for any \( x, y \in \mathbb{R} \). Prove that
a) \( f(nx) = nf(x) \) for any \( n \in \mathbb{Z} \), and any \( x \in \mathbb{R} \).
b) \( f(n) = nf(1) \) for any \( n \in \mathbb{Z} \).
c) \( f \left( \frac{1}{q} \right) = \frac{1}{q} f(1) \), for any \( q \in \mathbb{Z}^* \).
d) \( f \left( \frac{p}{q} \right) = \frac{p}{q} f(1) \), for any \( p \in \mathbb{Z} \) and any \( q \in \mathbb{Z}^* \).
e) If \( f \) is continuous in a point, then \( f(x) = xf(1) \), for any \( x \in \mathbb{R} \).
f) If \( f \) is monotonic, then \( f(x) = xf(1) \), for any \( x \in \mathbb{R} \).
g) If \( f \) is also multiplicative, i.e. \( f(xy) = f(x)f(y) \), then \( f(x) = x \), for any \( x \in \mathbb{R} \), or \( f(x) = 0 \) for any \( x \in \mathbb{R} \).

Solution: a) First remark \( f(0) = 0 \). Induction for \( n \in \mathbb{N} \), then use \( f(-x) = -f(x) \).
b) \( x = 1 \) in a).
c) \( n = q, \ x = \frac{1}{q} \) in a).
d) \( n = p, \ x = \frac{1}{q} \) in a), then use c).
e) It is easy to see that \( f \) is continuous in an point. Indeed if \( f \) is continuous in \( a \), then \( \lim_{x \to x_0} f(x) - f(x_0) = \lim_{x \to x_0} [f(x - x_0 + a) - f(a)] = f(a) - f(a) = 0 \). From d), \( f(r) = rf(1) \) for any \( r \in \mathbb{Q} \). For any \( x \in \mathbb{R} \), there is a sequence \( r_n \) of rationals convergent to \( x \), and then \( f(x) = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} r_n f(1) = xf(1) \).
f) For any real \( x \) there are sequences \( r_n \) increasing toward \( x \), and \( s_n \) decreasing toward \( x \). Then \( r_n f(1) = f(r_n) \leq f(x) \leq f(s_n) = s_n f(1) \) and we pass at the limit with \( n \to \infty \).
g) Let \( x \leq y \). Then \( f(y) = f(x) + f(y-x) = f(x) + f^2(\sqrt{y-x}) \geq f(x) \). The function \( f \) is increasing, so from f) we have \( f(x) = xf(1) \). But \( f(1) = f^2(1) \).

Problem 15.2. Find the continuous functions \( f : D \to \mathbb{R} \) satisfying
a) \( f(x + y) = f(x)f(y) \), for any \( x, y \in D = \mathbb{R} \).
b) \( f(xy) = f(x) + f(y) \), for any \( x, y \in D = (0, \infty) \).
c) \( f(xy) = f(x)f(y) \), for any \( x, y \in D = (0, \infty) \).

Solution: a) \( f(2x) = f^2(x) \geq 0 \), for any \( x \). If there is a \( x_0 \) such that \( f(x_0) = 0 \), then \( f(x) = f(x - x_0)f(x_0) = 0 \), for any \( x \). Therefore one solution is \( f \equiv 0 \), and to find the others we can suppose \( f(x) > 0 \), for any \( x \). The function \( g(x) = \ln f(x) \) is
continuous and satisfies \( g(x + y) = g(x) + g(y) \), for any \( x, y \in \mathbb{R} \). Using the problem

15.1. \( g(x) = xg(1) \). Then \( f(x) = e^{xg(1)} = a^x \), for any \( x \in \mathbb{R} \).

b) The function \( h : \mathbb{R} \to \mathbb{R} \), defined by \( h(u) = f(e^u) \) is continuous and satisfies

\[ h(u + v) = h(u) + h(v). \]

Using the problem 15.1 \( h(u) = uh(1) \), so \( f(x) = h(\ln x) = h(1) \ln x \).

c) The function \( k : \mathbb{R} \to \mathbb{R} \), defined by \( k(v) = f(e^v) \) is continuous and satisfies

\[ k(u + v) = f(e^u e^v) = f(e^u) f(e^v) = k(u) k(v). \]

Use a).

**Problem 15.3.** Find all continuous real-valued functions of one real variable \( f(x) \) such that \( f(\sqrt{x^2 + y^2}) = f(x) + f(y) \), for all real \( x \) and \( y \).

**Solution:** One can easily see that \( f \) must be even. Let \( g : [0, \infty) \to \mathbb{R} \), be the function defined by \( g(x) = f(\sqrt{x}) \). Then \( g \) is continuous and satisfies the Cauchy functional equation \( g(x + y) = g(x) + g(y) \), so there is a constant \( c \in \mathbb{R} \) such that 

\[ g(x) = cx. \]

Hence \( f(x) = cx^2 \), for any positive \( x \) and since \( f \) is even this formula holds for any real \( x \).

**Problem 15.4.** Find all continuous real-valued functions \( f(x) \) of one real variable such that \( f(\sqrt{x^2 + y^2}) = f(x)f(y) \) for all real \( x \) and \( y \).

**Solution:** The function \( f = 0 \) is a solution. To find other solutions, suppose there is a real \( x_0 \) such that \( f(x_0) \neq 0 \). From \( f(-x)f(x_0) = f(\sqrt{x^2 + x_0^2}) = f(x)f(x_0) \) we see \( f \) is an even function. The equality \( f(x) = f(\frac{x}{\sqrt{2}}) f(\frac{x}{\sqrt{2}}) \) shows that \( f(x) \geq 0 \) for any positive \( x \). Suppose there is a real \( x_1 \) such that \( f(x_1) = 0 \). Then using repeatedly the equality \( f(x) = f(\frac{x}{\sqrt{2}}) f(\frac{x}{\sqrt{2}}) \) we obtain \( f(\frac{x_1}{(\sqrt{2})^n}) = 0 \).

But \( f \) is continuous, hence \( f(0) = 0 \) and consequently \( f(|x_0|) = f(|x_0|) f(0) \). The contradiction that we obtained proves that \( f \) is taking only positive values, so one can define \( g : [0, \infty) \to \mathbb{R} \), \( g(x) = \ln f(\sqrt{x}) \). The continuous function \( g \) satisfies the Cauchy functional equation \( g(x + y) = g(x) + g(y) \), thus there is a constant \( c \) such that \( g(x) = cx \). Therefore \( f(x) = e^{cx^2} \), for any real \( x \) (we use that \( f \) is even). In conclusion the solutions of the equation are the functions \( f_a(x) = a^{x^2} \), with \( a \geq 0 \).

**Problem 15.5.** Determine all continuous functions \( f(x) \) such that \( f(x) + f(y) = f(\frac{x + y}{1 - xy}) \) for all real \( x \) and \( y \) with \( xy \neq 1 \).

**Solution:** Consider \( g(x) = f(\tan x) \). Then the continuous function \( g \) satisfies 

\[ g(x) + g(y) = g(x + y). \]

So there is a constant \( c \) such that \( g(x) = cx \) and therefore \( f(x) = c \arctan x \).

**Problem 15.6.** Determine the continuous function \( f \) such that \( f(x + y) = \frac{f(x) + f(y)}{1 + f(x)f(y)} \), for all reals \( x, y \).
Solution: We have
\[
1 + f(x + y) = \frac{(1 + f(x))(1 + f(y))}{(1 + f(x))(1 + f(y))}
\]
\[
1 - f(x + y) = \frac{(1 - f(x))(1 - f(y))}{1 + f(x)f(y)}
\]
\[
\frac{1 - f(x + y)}{1 + f(x + y)} = \frac{1 - f(x)}{1 + f(x)} \cdot \frac{1 - f(y)}{1 + f(y)}
\]
Therefore the continuous function \(g(x) = \frac{1 - f(x)}{1 + f(x)}\) satisfies the equation \(g(x + y) = g(x)g(y)\), so \(g(x) = a^x\) for \(a > 0\). It follows that \(f(x) = \frac{1 - a^x}{1 + a^x}\).

Problem 15.7. Determine the continuous function \(f : \mathbb{R} \to \mathbb{R}\) such that \(f(x + y) = f(x) + f(y) - 2f(x)f(y)\).

Solution: The relation can be written \(g(x+y) = g(x)g(y)\), where \(g(x) = 1 - 2f(x)\). It follows \(g(x) = a^x\) for some \(a > 0\) and then \(f(x) = \frac{1 - a^x}{2}\).

Problem 15.8. Determine the continuous function \(f : \mathbb{R} \to \mathbb{R}\) such that \(f(x + y) = f(x) + f(y) + 2xy - 1\).

Solution: Taking \(g(x) = f(x) - x^2 + 1\), we have \(g(x + y) = g(x) + g(y)\). Hence \(f(x) = x^2 + cx - 1\), for a constant \(c\).

Problem 15.9. Determine \(f\) continuous such that
\[f(x + y) = f(x) + f(y) + 3xy(x + y + 6) - 8\]

Solution: For \(g(x) = f(x) - x^3 - 9x^2 - 8\), we have \(g(x + y) = g(x) + g(y)\), etc.

2. Functions over \(\mathbb{R}\)

Problem 15.10. Let \(f : \mathbb{R} \to \mathbb{R}\) be a function with the property
\[f^3(x + y) + f^3(x - y) = (f(x) + f(y))^3 + (f(x) - f(y))^3\]
for all \(x, y \in \mathbb{R}\). Show that \(f(x + y) = f(x) + f(y)\) for all \(x, y \in \mathbb{R}\).

Problem 15.11. Find the functions \(f : (0, \frac{\pi}{2}) \to \mathbb{R}\) such that
\[f(x) = x \tan \left(\frac{f(x)}{\tan x}\right)\]

Solution: The relation can be written
\[
\tan x = \frac{\tan \left(\frac{f(x)}{\tan x}\right)}{\frac{f(x)}{\tan x}}
\]
Since the function \(g(x) = \frac{\tan x}{x}\) is injective over the interval \((0, \frac{\pi}{2})\) it follows that \(\frac{f(x)}{\tan x} = x\).
Problem 15.12. Find all real valued functions such that \( f(x) + 2xf(1-x) = 2x + 1 \).

**Solution:** The substitution \( x \to 1 - x \) gives \( f(1-x) + (2 - 2x)f(x) = 3 - 2x \). Solving the system in \( f(x) \) and \( f(1-x) \) formed by this equation and the original one we obtain \( (2x - 1)^2 f(x) = (2x - 1)^2 \), for any real \( x \). Therefore \( f(x) = 1 \) for \( x \neq \frac{1}{2} \). But taking \( x = \frac{1}{2} \) in the original equation we obtain \( f(\frac{1}{2}) = 1 \), so the only solution is \( f(x) = 1 \) for any \( x \).

Problem 15.13. Find all real valued functions \( f(x) \) defined for all \( x \neq 0, 1 \) which satisfy the functional equation \( f(x) + f(1-x) = 1 + x \).

**Solution:** If \( g(x) = 1 - x^{-1} \), then \( g(g(x)) = x \) for all \( x \neq 0, 1 \). The functional equation is the first equation of the following system and substituting \( x \) first with \( g(x) \) and also with \( g(g(x)) \) we get the second and the third equation of the system

\[
\begin{align*}
  f(x) & + f(g(x)) = 1 + x \\
  f(g(x)) & + f(g(g(x))) = 1 + g(x) \\
  f(g(g(x))) & + f(x) = 1 + g(g(x))
\end{align*}
\]

Adding the first and third equation and subtracting the second one gives \( f(x) = \frac{1 + x - g(x) + g(g(x))}{2} \) and a simple computation shows that this \( f \) is a solution.

Problem 15.14. Let \( a \) be a fixed real number. Find the functions \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(0) = \frac{1}{2} \), and satisfying for any \( x, y \in \mathbb{R} \)

\[
(2.1) \quad f(x+y) = f(x)f(a-y) + f(y)f(a-x)
\]

**Solution:** For \( x = y = 0 \), we get \( f(a) = \frac{1}{2} \). Taking \( y = 0 \) in the relation 2.1 we have \( f(x) = f(a-x) \). Using this and taking \( y = a \) in 2.1 we obtain \( f(x+a) = f(x) \).

As a consequence \( f(x) = f(-x) \). Now with \( y = -x \), \( f(0) = 2f(x) \), so \( f^4(x) = \frac{1}{4} \).

But replacing \( f(a-x) = f(x) \) in 2.1 we have \( f(x+y) = 2f(x)f(y) \), which for \( y = x \) proves \( f(2x) = 2f^2(x) \geq 0 \). Hence \( f(x) = \frac{1}{2} \).

Problem 15.15. Find all the functions \( f : (0, \infty) \to (0, \infty) \) satisfying the conditions

i) \( f(xf(y)) = yf(x) \), for all \( x, y > 0 \)

ii) \( \lim_{x \to \infty} f(x) = 0 \)

**Solution:** Taking \( x = 1 \) in i) we get \( f(f(y)) = yf(1) \). We compute in two ways

\[
\begin{align*}
  f(f(xf(y))) & = xf(y)f(1) \\
  f(f(xf(y))) & = f(yf(x)) = xf(y) \text{ using i) twice.}
\end{align*}
\]

Then \( f(1) = 1 \), since \( f \) cannot take the value 0, and consequently \( f(f(y)) = y \).

Taking \( y = f(z) \) in i) gives \( f(xz) = f(x)f(z) \) for all \( x, z > 0 \).

In particular \( f(x^n) = f(x)^n \). For a fixed \( x > 1 \) taking the limit with \( n \to \infty \) we have \( x^n \to \infty \) and get \( 0 = \lim_{n \to \infty} f(x^n) = \lim_{n \to \infty} f(x)^n \). Therefore \( f(x) < 1 \), which with the equation \( f(xz) = f(x)f(z) \) proves \( f \) is strictly decreasing.
The decreasing function \( g : \mathbb{R} \to \mathbb{R} \), \( g(u) = \ln f(e^u) \) satisfies the equation 
\[ g(u + v) = g(u) + g(v). \]
For any rational \( u \), \( g(u) = ug(1) \). For a fixed real \( u \) let \( u_n \), respectively \( v_n \), be an increasing, respectively decreasing, sequence of rationals such that \( u_n \leq u \leq v_n \) and \( \lim u_n = \lim v_n = u \). Then \( u_ng(1) = g(u_n) \geq g(u) \geq g(v_n) = v_ng(1) \), and passing at the limit with \( n \to \infty \), \( g(u) = ug(1) \).

Hence \( f(x) = x^g(1) \). Since \( f \) is decreasing, \( g(1) < 0 \). The relation \( f(f(x)) = x \), gives \( g'(1) = 1 \), with the only convenient solution \( g(1) = -1 \). Therefore the only function satisfying the requirements is \( f(x) = \frac{1}{x} \).

\textbf{Problem 15.16.} Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that \( \sqrt{f(x)g(y)} = f(\sqrt{xy}) \) for any \( x, y \in \mathbb{R} \). Prove that \( \sqrt{f(x_1)f(x_2)\ldots f(x_n)} = f(\sqrt{x_1x_2\ldots x_n}) \), for any \( x_1, x_2, \ldots, x_n \in \mathbb{R} \).

\textbf{Solution :} Let denote by \( P(n) \) the requirement of the problem. By induction it is easy to prove \( P(n) \) for \( n = 2^k \). To end the proof it is enough to show that from \( P(n) \) it follows \( P(n-1) \). And for this take \( x_n = n\sqrt{x_1x_2\ldots x_{n-1}} \) in \( P(n) \).

\textbf{Problem 15.17.} Let \( f : \mathbb{R} \to (0, \infty) \) be a function with the properties
1. \( f(x) = 0 \) if and only if \( x = 0 \)
2. \( f(x+y) \leq f(x) + f(y) \) for any \( x, y \)
3. \( f(xy) = f(x)f(y) \) for any \( x, y \)
4. \( f \) monotonic on \((0, \infty)\)
5. \( f(2) = 2 \)
Then \( f(x) = |x| \).

\textbf{Solution :} From \( f(1) = f(1)f(1) \) and property 1. we obtain \( f(1) = 1 \). Also the equation \( f(-1)f(-1) = f(1) = 1 \) gives \( f(-1) = 1 \). This with the multiplicativity proves that \( f \) is an even function. We prove by induction that \( f(n) = n \) for any positive integer \( n \). For \( n = 1 \) and \( n = 2 \) the statement holds. Suppose that \( f(k) = k \) for any \( k \leq n \). Then \( f(n+1) \leq f(n) + f(1) = n + 1 \). For \( n = 2k \), \( f(n+1) = f(2k+1) \geq f(2k) = f(1) + f(k) = f(k) + f(1) = n + 1 \). For \( n = 2k + 1 \), \( f(n+1) = f(2k+1) = f(2)f(k+1) = f(2)f(k) = n + 1 \). In this way using also the fact that \( f \) is even we proved \( f(x) = |x| \) for any \( x \in \mathbb{Z} \).

From \( 1 = f(1)f(\frac{1}{m}) \) we get \( f(\frac{1}{m}) = \frac{1}{m} \) and \( f(m)f(\frac{1}{m}) = f(\frac{1}{m}) = \frac{m}{m} \). We proved \( f(x) = |x| \) for \( x \in \mathbb{Q} \).

Let \( x \) be a positive real number. There are an increasing sequence \((a_n)_n \) and a decreasing sequence \((b_n)_n \) of positive rational numbers, both convergent to \( x \) and such that \( a_n < x < b_n \). Suppose \( f \) is increasing (the case decreasing is similar). Then \( a_n = f(a_n) \leq f(x) \leq f(b_n) = b_n \). Passing at limit with \( n \to \infty \) we have \( f(x) = x \). Since \( f \) is even \( f(x) = |x| \) for any real number \( x \).

\textbf{Problem 15.18.} Two functions \( f, g : \mathbb{R} \to \mathbb{R} \) are said to be similar if there is a bijection \( h : \mathbb{R} \to \mathbb{R} \) such that \( h \circ f = g \circ h \). Prove that the functions \( \sin \) and \( \cos \) are not similar.

\textbf{Solution :} Suppose there is \( h : \mathbb{R} \to \mathbb{R} \) such that \( h(\sin x) = \cos(h(x)) \) for all \( x \in \mathbb{R} \). Then for any \( y \in [-1, 1] \) there is \( x \in [-1, 1] \) such that \( h(x) = y \). Indeed, there is \( u \in \mathbb{R} \) such that \( \cos u = y \), and there is \( v \in \mathbb{R} \) such that \( h(v) = u \). Taking \( x = \sin v \) we have \( h(x) = h(\sin v) = \cos(h(v)) = \cos u = y \). Consider \( x_1, x_2 \in [-1, 1] \) such
that \( h(x_1) = -1 \) and \( h(x_2) = 1 \). Then \( h(\sin x_1) = \cos(-1) = \cos 1 = h(\sin x_2) \) and since \( h \) is injective \( \sin x_1 = \sin x_2 \). But \( \sin \) is injective on \([-1, 1]\), so \( x_1 = x_2 \). Contradiction with \( h(x_1) \neq h(x_2) \).

**Problem 15.19.** Prove that there exists a unique function \( f \) from the set \( \mathbb{R}^+ \) of positive real numbers to \( \mathbb{R}^+ \) such that \( f(f(x)) = 6x - f(x) \) and \( f(x) > 0 \) for all \( x > 0 \). [P1988]

**Solution:** Fix \( x > 0 \). We apply \( f^{(n)} = f \circ f \circ \ldots \circ f \) to the relation \( f(f(x)) = 6x - f(x) \) and we obtain

\[
f(a_n f(x) + b_n) = c_n f(x) + d_n x \quad (*).
\]

Of course \( a_0 = 1, b_0 = 0, c_0 = -1, d_0 = 6 \). Applying \( f \) once more we get

\[
f(c_n f(x) + d_n x) = f(f(a_n f(x) + b_n x)) = (6a_n - c_n)f(x) + (6b_n - d_n)x.
\]

Therefore \( a_{n+1} = c_n, b_{n+1} = d_n, c_{n+1} = 6a_n - c_n, d_{n+1} = 6b_n - d_n \). Solving the system we have \( c_n = \frac{1}{5}(2^{n+2} - (-3)^{n+2}) \) and \( d_n = \frac{6}{5}(2^{n+1} - (-3)^{n+1}) \). The inequality \( c_n f(x) + d_n x > 0 \) which is a consequence of (*), implies for \( n = 2k - 2 \) the left (respectively, the right) side of the inequality

\[
\frac{2}{1 + (2/3)^{2k+1}} < \frac{f(x)}{x} < \frac{2(1 + (2/3)^{2k-1})}{1 - (2/3)^{2k}}
\]

Passing at the limit with \( k \to \infty \) we obtain \( f(x) = 2x \) which is the unique solution.

**Problem 15.20.** Let \( n \) be a fixed integer \( \geq 2 \). Determine all functions \( f(x) \), which are bounded for \( 0 < x < a \), and which satisfy the functional equation

\[
f(x) = \frac{1}{n^2} \left( f\left( \frac{x}{n} \right) + f\left( \frac{x + a}{n} \right) + \ldots + f\left( \frac{x + (n-1)a}{n} \right) \right)
\]

**Solution:** The function \( f \) satisfies the equation

\[
f(x) = \frac{1}{n^2} \sum_{k=0}^{n-1} f\left( \frac{x + ka}{n} \right)
\]

Then, for any \( k = 1, n-1 \) we have \( f\left( \frac{x + ka}{n} \right) = \frac{1}{n^2} \sum_{j=0}^{n-1} f\left( \frac{x + ka + ja}{n} \right) \). Replacing this into the equation 2.2 we get

\[
f(x) = \frac{1}{n^4} \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} f\left( \frac{x + ka + ja}{n^2} \right) = \frac{1}{n^4} \sum_{k=0}^{n^2-1} f\left( \frac{x + ja}{n^2} \right)
\]

By induction one can show that for any integer \( p \geq 1 \)

\[
f(x) = \frac{1}{n^{2p}} \sum_{k=0}^{n^p-1} f\left( \frac{x + ka}{n^p} \right)
\]

Let \( M \) be a real number such that \( |f(x)| \leq M \) for any \( x \in [0, a] \). Then for any \( p, |f(x)| \leq \frac{Mn^p}{n^{2p}} \). Passing at the limit with \( p \to \infty \) we see that necessarily \( f(x) = 0 \).

**Problem 15.21.** Let \( a_0, a_1, \ldots, a_n \) be positive real numbers such that \( a_0 + a_1 x + \ldots + a_n x^n > 0 \) for all \( x \in \mathbb{R} \). Show that there is no function \( f : \mathbb{R} \to \mathbb{R} \), such that \( a_0 + a_1 f + \ldots + a_n (f \circ f \circ \ldots \circ f) = 0 \).
3. Functions over $\mathbb{Z}$ or $\mathbb{N}$

**Problem 15.22.** How many functions $f : \mathbb{N} \to \mathbb{N}$ satisfy $f(f(n)) = n + 1$ for all $n \in \mathbb{N}$?

**Solution:** Using the associativity of the composition of functions, we have

$$f(n + 1) = [(f \circ (f \circ f))(n) = [(f \circ f) \circ f](n) = f(n) + 1, \forall n \in \mathbb{N}$$

From here it follows that $f(n) = f(0) + n$, for all $n$. Then $f(f(n)) = f(f(0) + n) = 2f(0) + n$, so $f(0) = \frac{1}{2}$. Contradiction, hence there is no such function.

**Problem 15.23.** Let $f$ be a weakly increasing function from the positive integers to the positive real numbers (i.e. $m < n$ implies $f(m) \leq f(n)$). Suppose that $f(m)f(n) = f(mn)$ whenever $m$ and $n$ are relatively prime numbers. Show that there is a nonnegative number $\alpha$ such that $f(n) = n^\alpha$.

**Problem 15.24.** Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions.

(i) $f(f(n)) = n$, for all integers $n$;
(ii) $f(f(n + 2) + 2) = n$ for all integers $n$;
(iii) $f(0) = 1$.

[P1992]

**Solution:** Using (ii) and (i) we get $f(n + 2) = f(n) - 2$. Therefore $f(2k) = f(0) - 2k = 1 - 2k$ and $f(2k + 1) = f(1) - 2k = f(0) - 2k = -2k$.

**Problem 15.25.** Determine, with proof, all the functions $f : \mathbb{Z} \to \mathbb{Z}$ which satisfy $f(x + f(y)) = f(x) + y$ for all integers $x, y$.

**Solution:** From $f(f(x + f(y))) = f(y + f(x)) = x + f(y)$ we deduce $f(f(z)) = z$, for all integer $z$. Using the original identity with $x = f(z)$ we obtain $f(f(z) + f(y)) = f(f(z)) + y = z + y$. We apply $f$ and we get $f(z) + f(y) = f(f(f(z) + f(y))) = f(z + y)$. This is the Cauchy functional equation which yields $f(x) = f(1)x$ for all integer $x$. Replacing this in the original equation we obtain $f(1)^2y = y$ for every $y$, so $f(1) = \pm 1$. Therefore the solutions are $f_1(x) = x$ and $f_2(x) = -x$.

**Problem 15.26.** A function $f : \mathbb{N} \to \mathbb{C}$ is called completely multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers $m$ and $n$. Find all completely multiplicative functions $f$ with the property that the function $F(n) = f(1) + f(2) + \ldots + f(n)$ is also completely multiplicative.

**Solution:** Subtracting the relations $F(n - 1)F(2) = F(2n - 2)$ and $F(n)F(2) = F(2n)$ and using the multiplicity of $f$ we get $f(n) = f(2n - 1)$ (*). Using successively this equation for $n = 2, 3, 5$ we obtain $f(2) = f(3) = f(5) = f(9) = f(3)^2$. We prove by induction that $f(n) = f(2)$, for any $n \geq 2$. Suppose this is true for all positive integers smaller than $n$. If $n$ is a prime number then is odd and by (*), $f(n) = f\left(\frac{n + 1}{2}\right) = f(2)$. If $n$ is not prime then there are $p, q$ positive integers such that $n = pq$. Then $f(n) = f(p)f(q) = f(2)^2 = f(2)$. Therefore the only functions satisfying the requirements are $f_1(1) = 1, f_1(n) = 0$, for $n \geq 2$ and $f_2(n) = 1$ for any $n$. 

4. Continuous functions

Problem 15.27. Let \( a \in (0,1) \) be a real number. Determine the continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(0) = a \), and \( f(x) - f(ax) = x \), for any real \( x \).

Problem 15.28. Let \( a, b \in \mathbb{R} \), \( |a| \neq 1 \). Determine the continuous function \( f : \mathbb{R} \to \mathbb{R} \) such that \( f(ax + b) = f(x) \).

Problem 15.29. Is there any continuous function \( f : \mathbb{R} \to \mathbb{R} \), satisfying the condition \( f(f(x)) = -x \)?

Solution: The answer is NO. Indeed if we suppose such function exists, then \( f \) must be injective and consequently strictly monotonous. But in this case \( f(f(x)) \) is an increasing function. Contradiction.

Problem 15.30. Solve the functional equation \( f(x) = f\left(\frac{x}{1-x}\right) \), \( x \neq 1 \), subject to the condition that \( f \) is continuous at \( x = 0 \).

Solution: Consider the function \( g(x) = \frac{x}{1-x} \). The composition of \( g \) with itself \( n \) times is given by \( g^{(n)}(x) = \frac{x}{1-nx} \), for all \( x \notin \{1, 1/2, 1/3, \ldots\} \). Passing at the limit with \( n \to \infty \) in the relation \( f(x) = f\left(\frac{x}{1-nx}\right) \), we get \( f(x) = f(0) \) for all \( x \notin \{1, 1/2, 1/3, \ldots\} \). But the functional equation also gives \( f(1) = f(1/2) = f(1/3) = \ldots \). So \( f(1/m) = f(1/n) \) for all \( m, n \). Passing at limit with \( n \to \infty \) in this relation we get \( f(1/m) = f(0) \). Therefore \( f(x) = f(0) \) for all real \( x \).

Problem 15.31. Find all continuous functions \( f \) such that \( f(2x+1) = f(x) \) for all real \( x \).

Solution: If \( g(x) = \frac{x-1}{2} \), then \( f(g(x)) = f(x) \) for all real \( x \). This implies \( f(g^{(n)}(x)) = f(x) \), for all real \( x \), where \( g^{(n)}(x) \) is the composition \( g \circ g \circ \ldots \circ g \) of \( g \) with itself \( n \) times. An easy induction argument shows that \( g^{(n)}(x) = \frac{x - 2^n + 1}{2^n} \). Passing at the limit with \( n \to \infty \) in the relation

\[
 f\left(\frac{x+1}{2^n} - 1\right) = f(x)
\]

we obtain \( f(-1) = f(x) \), for any real \( x \). So \( f \) is a constant function.

Problem 15.32. For which real numbers \( k \) does there exist a continuous real-valued function \( f(x) \) such that \( f(f(x)) = kx^3 \) for all real \( x \)?

Solution: The answer is \( k \geq 0 \). Indeed, for \( k \geq 0 \), the function \( f(x) = \sqrt[3]{k}x^3 \) is a solution of the equation. Let \( k < 0 \) and suppose there is \( f \) satisfying the equation. Then \( f \) must be injective, since \( x \mapsto kx^3 \) is. But \( f \) is also continuous, so \( f \) must be monotonic. Then \( f \circ f \) is an increasing function, which contradicts the fact that \( x \mapsto kx^3 \) is decreasing.
5. Differentiable functions

**Problem 15.33.** Find all the functions $f$ differentiable in 0 with the property $f(x + y) = f(x)f(y)$ for all real numbers $x, y$.

**Solution:** Taking $x = y = 0$ in the equation gives $f(0)^2 = f(0)$. If $f(0) = 0$, then $f(x) = f(x)f(0) = 0$, for all $x$. Suppose $f(0) = 1$ and let $f'(0) = a$. Then

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} = af(x)$$

for all real numbers $x$. Solving this differential equation $f(x) = e^{ax}$.

**Problem 15.34.** Find all functions $f$ that satisfy the equation

$$\left( \int f(x)dx \right) \left( \int \frac{1}{f(x)}dx \right) = -1$$

**Solution:** With $F(x) = \int f(x)dx$ the equation writes $\int \frac{1}{f(x)}dx = -\frac{1}{F(x)}$. Taking the derivative gives $f^2(x) = F^2(x)$. We derive again and since $f(x) \neq 0$, for any $x$, we get $F(x) = f'(x) = F''(x)$. There are reals $a, b$ such that $F(x) = ae^x + be^{-x}$, and replacing in $f^2(x) = F^2(x)$ we see that $ab = 0$. The solutions are then $f_1(x) = ae^x, f_2(x) = ce^{\frac{1}{a}}$.

**Problem 15.35.** Find all twice-differentiable functions $f(x)$ on the real line which satisfy $f(x + y)f(x - y) = f(x)^2 - f(y)^2$ for all real $x$ and $y$.

**Solution:** The following things are easy consequences of the equation:

1. $f(0) = 0$
2. $f$ is odd
3. $f'(x + y)f(x - y) + f(x + y)f'(x - y) = 2f(x)f'(x)$, for any $x, y$
4. $f''(x + y)f(x - y) - f(x + y)f''(x - y) = 0$, for any $x, y$
5. If there is $z$ such that $f(z) \neq 0$ then $f''(x) = \frac{f''(z)}{f(z)}f(x)$, for any $x$.
6. We solve this differential equation and we get $f(x) = \sin ax$.

**Problem 15.36.** Suppose $f$ and $g$ are non-constant, differentiable, real-valued functions defined on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers $x$ and $y$,

$$f(x + y) = f(x)f(y) - g(x)g(y),$$

$$g(x + y) = f(x)g(y) + g(x)f(y).$$

If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all $x$. [P1991]

**Solution:** With $x = y = 0$ in the two identities we get $f(0) = f(0)^2 - g(0)^2$, $g(0) = 2f(0)g(0)$, which implies $g(0) = 0$. On the other side $g(x) = f(x)g(0) + g(x)f(0)$, hence $f(0) = 1$. Taking the derivative with respect to $x$

$$f'(x + y) = f'(x)f(y) - g'(x)g(y),$$

$$g'(x + y) = f'(x)g(y) + g'(x)f(y).$$
For $x = 0$ we obtain
\[ f'(y) = -ag(y), \]
\[ g'(y) = af(y). \]
Solving this system with the given initial values we obtain $f(x) = \cos ax$ and $g(x) = \sin ax$. Therefore $f^2 + g^2 = 1$.

**Problem 15.37.** If $f$ is a differentiable function such that $\int_0^x f(t) dt = f^2(x)$ for all $x$, find $f$.

**Problem 15.38.** A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval $(a, b)$ and a nonzero function $g$ defined on $(a, b)$ such that this wrong product rule is true for $x$ in $(a, b)$. [P1988]

**Solution:** The equation is equivalent with $2xg(x) = (2x - 1)g'(x)$ and $g(x) = e^x\sqrt{|2x - 1|}$ is a solution on any open interval which doesn’t contain $\frac{1}{2}$.

**Problem 15.39.** For all real $x$, the real-valued function $y = f(x)$ satisfies $y'' - 2y' + y = 2e^x$.

(a) If $f(x) > 0$ for all real $x$, must $f'(x) > 0$ for all real $x$? Explain.

(b) If $f'(x) > 0$ for all real $x$, must $f(x) > 0$ for all real $x$? Explain. [P1987]

**Solution:** The general solution of the 2-order differential equation is $f(x) = ae^{x^2} + bxe^x + x^2e^x$ with $f'(x) = [x^2 + (b + 2)x + (a + b)]e^x$. The condition $f(x) > 0$ for all real $x$ is equivalent to $\Delta = b^2 - 4a < 0$ and the condition $f'(x) > 0$ for all real $x$ is equivalent to $(b + 2)^2 - 4(a + b) = \Delta + 4 < 0$. The answer to (a) is no and the answer to (b) is yes.

**Problem 15.40.** Find all real-valued continuously differentiable functions $f$ on the real line such that for all $x$,
\[ (f(x))^2 = \int_0^x [(f(t))^2 + (f'(t))^2] dt + 1990. \]

[P1990]

**Solution:** Differentiating we get $2f'(x)f(x) = (f(x))^2 + (f'(x))^2$, so $f'(x) = f(x)$. We introduce the solution $f(x) = ce^x$ of this differential equation into the original equation and we have $c^2e^{2x} = c^2(e^{2x} - 1) + 1990$. Thus $c = \sqrt{1990}$ and $f(x) = \sqrt{1990}e^x$.

6. Functional inequalities

**Problem 15.41.** Determine the functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) - f(y) \leq x - y$ for all $x, y \in \mathbb{R}$.

**Solution:** The condition in the problem can be written $f(x) - x \leq f(y) - y$, for all $x, y$. Changing $x$ with $y$ we get the opposite inequality, therefore $f(x) - x$ is constant and then $f(x) = x + c$. 
Problem 15.42. Determine the functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) satisfying \( f(x + y) \leq f(xy) \) for all \( x, y \in \mathbb{R} \).

Solution: Taking \( y = 0 \) we have \( f(x) \leq f(0) \) for all \( x \in \mathbb{R} \). Also for \( x = -y = \sqrt{t} \) we get \( f(0) \leq f(-t) \), so \( f(x) = f(0) \) for \( x \leq 0 \). If \( y = -1 \), we obtain \( f(x - 1) \leq f(-x) \). Changing \( x \) into \( -x \) it leads to \( f(-x - 1) \leq f(x) \) for all \( x \in \mathbb{R} \). If \( x > 0 \), then \( -x - 1 < 0 \) so \( f(0) = f(-x - 1) \leq f(x) \leq f(0) \) and as a consequence \( f(x) = f(0) \) for all real \( x \).

Problem 15.43. Find the functions \( f : (0, \infty) \rightarrow (0, 1] \) such that \( f(xy) \leq f(x)f(y) \) for all \( x, y > 0 \).

Solution: Taking \( x = y = 1 \) we have \( f(1) \leq f(1)^2 \), so \( 1 \leq f(1) \). But \( f(1) \leq 1 \) by the way the function is defined, hence \( f(1) = 1 \).

We take \( y = \frac{1}{x} \) and we have \( 1 = f(1) \leq f(x)f\left(\frac{1}{x}\right) \leq f(x) \leq 1 \). Therefore \( f(x) = 1 \) for all \( x > 0 \).

Problem 15.44. Determine all the functions \( f : \mathbb{N} \rightarrow \mathbb{R} \) satisfying for all \( k, m, n \in \mathbb{N} \) the inequality \( f(kn) + f(km) - f(k)f(mn) \geq 1 \).

Solution: For \( k = m = n = 0 \), \( 2f(0) - f(0)^2 \geq 1 \), so \( f(0) = 1 \). Also for \( k = m = n = 1 \) we obtain \( f(1) = 1 \). Taking \( k = 0 \) gives \( f(mn) \leq 1 \) for all \( m, n \). For \( m = 0 \) in the original inequality we obtain \( f(kn) \geq f(k) \). Therefore, \( 1 = f(1) \leq f(n \cdot 1) \leq 1 \), and \( f(n) = 1 \) for any \( n \).

Problem 15.45. Determine all the functions \( f : \mathbb{R} \rightarrow \mathbb{R} \) satisfying \( f(x - y) - xf(y) \leq 1 - x \), for all \( x, y \in \mathbb{R} \).

Solution: Taking \( x = 0 \) and \( y = -y \) we see that \( f(y) \leq 1 \) for all \( y \in \mathbb{R} \).

For \( x = 2y \) we obtain \((1 - 2y)(1 - f(y)) \geq 0 \), for all \( y \), so \( f(y) \geq 1 \) for \( y \geq \frac{1}{2} \).

Take \( x = y + 1 \) and arrange the inequality as \((y + 1)(1 - f(y)) \leq 0 \). Therefore \( f(y) = 1 \) for \( y > -1 \).

Take \( x = 1 \). For \( y < 2 \) we have \( x - y > -1 \), so \( f(x - y) = 1 \). The inequality becomes \( f(y) \leq 1 \). Therefore \( f(y) = 1 \), for all \( y \).

Problem 15.46. If \( f : \mathbb{R} \rightarrow \mathbb{R} \) is an increasing bijection, find the functions \( g : \mathbb{R} \rightarrow \mathbb{R} \) satisfying \((f \circ g)(x) \leq x \leq (g \circ f)(x)\).

Solution: Since \( f \) is increasing, the inequality \( f(g(x)) \leq x = f(f^{-1}(x)) \) gives \( g(x) \leq f^{-1}(x) \). But also \( f^{-1}(x) \leq g(f(f^{-1}(x))) = g(x) \), so \( g = f^{-1} \).

Problem 15.47. Is there an injective function \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that
\[ f(x^2) - f(x)^2 \geq \frac{1}{4} \]

Problem 15.48. Determine the convex functions \( f : [0, 1] \rightarrow \mathbb{R} \) such that
\[ 2f(1/2) = f(0) + f(1) \]

Problem 15.49. Determine the function \( f : \mathbb{R}^* \rightarrow (0, 1] \) such that \( f(xy) \leq f(x)f(y) \).
Problem 15.50. Find the real valued function $f$ such that $f^3(x) + 1 \leq x \leq f(x^3 + 1)$, $\forall x \in \mathbb{R}$.
CHAPTER 16

Sequences

1. General Term

**Problem 16.1.** Fibonacci posed the following problem: Suppose that rabbits live forever and that every month each pair produces a new pair which becomes productive at age 2 months. If we start with one newborn pair, how many pairs of rabbits will we have in the \( n \)-th month?

**Solution:** Denote by \( f_n \) the number of pairs of rabbits in the \( n \)-th month. Then \( f_0 = 1 \) and \( f_1 = 1 \). We remark that in the \((n + 1)\)-th month we will have all the pairs from the \( n \)-th month plus a newborn pair for each pair who was born earlier than the \( n \)-th month. So \( f_{n+1} = f_n + f_{n-1} \). Solving this second order recurrence gives

\[
f_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right]
\]

**Problem 16.2.** Let \( T_0 = 2, T_1 = 3, T_2 = 6 \), and for \( n \geq 3 \),

\[
T_n = (n + 4)T_{n-1} + 4nT_{n-2} + (4n - 8)T_{n-3}.
\]

The first few terms are

\[2, 3, 6, 14, 40, 152, 784, 5168, 40576,\]

Find, with proof, a formula for \( T_n \) of the form \( T_n = A_n + B_n \), where \( \{A_n\} \) and \( \{B_n\} \) are well-known sequences. [P1990]

**Solution:** The characteristic equation of the recurrence can be arranged as \( t^3 - 4t^2 + 8 = n(t^2 - 4t + 4) \). One can see that \( t = 2 \) is a zero of the polynomials from both sides of this equality. This shows that a geometric sequence of ratio 2 satisfies the recurrence, which makes us to guess that \( A_n = 2^n \). Then the first four terms of the sequence \( (B_n)_n \) are \( B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 6, B_4 = 24 \) which allows us to see that \( B_n = n! \).

**Problem 16.3.** The sequence \( (a_n)_{n \geq 1} \) is defined by \( a_1 = 1, a_2 = 2, a_3 = 24 \), and, for \( n \geq 4 \),

\[
a_n = \frac{6a_{n-1}a_{n-3} - 8a_{n-1}a_{n-2}}{a_{n-2}a_{n-3}}.
\]

Show that, for all \( n \), \( a_n \) is an integer multiple of \( n \). [P1999]
Solution: Let \( b_n = \frac{a_n}{a_{n-1}} \) which satisfies \( b_n = 6b_{n-1} - 8b_{n-2} \); with the initial conditions \( b_2 = 2, b_3 = 12 \), one easily obtains \( b_n = 2^{n-1}(2^{n-2} - 1) \), and so

\[
a_n = 2^{n(n-1)/2} \prod_{i=1}^{n-1} (2^i - 1).
\]

To see that \( n \) divides \( a_n \), factor \( n \) as \( 2^k m \), with \( m \) odd. Then note that \( k \leq n \leq n(n-1)/2 \), and that there exists \( i \leq m-1 \) such that \( m \) divides \( 2^i - 1 \), namely \( i = \phi(m) \) (Euler’s totient function: the number of integers in \( \{1, \ldots, m\} \) relatively prime to \( m \)).

Second solution: Let \( n = 2^k \prod_{i=1}^s p_i^{k_i} \) be the factorisation of \( n \) in prime factors. Since \( k \leq \frac{n}{2} \leq \frac{n(n-1)}{2} \), \( 2^k \) divides \( a_n \). By Fermat’s theorem \( p_i \) divides \( 2^{(p_i-1)/2} - 1 \) and \( k_i(p_i - 1) < p_i^{k_i} \leq n \), so \( p_i^{k_i} \) divides \( (2^{(p_i-1)/2} - 1)(2^{(p_i-1)/2} - 1) \ldots (2^{(p_i-1)/2} - 1) \) which divides \( a_n \).

**Problem 16.4.** Let \((x_n)_{n \geq 0}\) be a sequence of nonzero real numbers such that \( x_n^2 - x_{n-1}x_{n+1} = 1 \) for \( n = 1, 2, 3, \ldots \). Prove there exists a real number \( a \) such that \( x_{n+1} = ax_n - x_{n-1} \) for all \( n \geq 1 \). [P1993]

**Solution:** For any \( n \in \mathbb{N} \), \( x_n^2 - x_{n-1}x_{n+1} = x_{n+1}^2 - x_nx_{n+2} \) which can be rewritten as \( \frac{x_{n+2} + x_n}{x_{n+1}} = \frac{x_{n+1} + x_{n-1}}{x_n} \). So \( x_{n+1} = ax_n - x_{n-1} \) with \( a = \frac{x_1 + x_3}{x_2} \).

**Problem 16.5.** Let \( a \geq 1 \) be an integer, and the sequence \( x_n \) defined by \( x_1 > 0 \), \( x_{n+1} = x_n \sqrt{1 + a^2 + a \sqrt{1 + x_n^2}} \). Prove that among the first \( n \) terms of the sequence there are at least \( \left[ \frac{n}{3} \right] \) irrational terms.

**Problem 16.6.** Let \((x_n)_{n \geq 0}\) be a sequence defined by \( x_{n+2} = ax_{n+1} + bx_n \), where \( a, b, x_1, x_2 \) are integers. If \( a^2 + 4b \) is not a perfect square, prove there are no three consecutive terms in the sequence such that \( x_{n+2} = x_{n+1}^2 \).

**Solution:** Suppose we have such terms, and denote by \( t \) the rational number \( \frac{x_{n+2}}{x_{n+1}} = \frac{x_{n+1}}{x_n} \). Then \( t = a + \frac{b}{t} \) or \( t^2 - at - b = 0 \). But the discriminant \( a^2 + 4b \) is not a perfect square, so \( t \) is irrational. Contradiction.

**Problem 16.7.** Let \((a_n)\) be a sequence defined by \( a_0 = 0 \), \( a_{n+1} = 1 + a_{[(n-1)/2]} \), for any \( n \geq 0 \). Prove that \( a_n = \lceil \log_2(n+1) \rceil \), \( \lceil x \rceil \) is the integer satisfying \( x - 1 < [x] \leq x \).

**Solution:** It suffices to prove that for \( 2^k \leq n+1 < 2^{k+1} \), we have \( a_n = k \). Suppose this is true for \( n = 1, 2, \ldots, m \) and we prove it for \( n = m + 1 \).

Let \( k \) be such that \( 2^k \leq m + 2 < 2^{k+1} \). Then \( 2^{k-1} \leq m + 1 < 2^k \). It follows that \( 2^{k-1} \leq \left[ \frac{m}{2} \right] + 1 < 2^k \), so by the induction hypothesis, \( a_{m+1} = 1 + a_{[m/2]} = k \).

**Problem 16.8.** Let \( k \geq 2 \) be a fixed integer. Determine the sequence \( a_n \) satisfying \( a_0 = 0 \) and \( a_n - a_{[n/k]} = 1 \).
Solve the second order recurrence for \( x \): 

\[
\begin{align*}
2x^2 - 4x + 2 &= 0 \\
x &= 1, x = 2 \\
2x + 2 &= 0 \\
x &= -1, x = -2
\end{align*}
\]

Since the second order recurrence is linear, we can solve it using the characteristic equation. The characteristic equation is:

\[
2x^2 - 4x + 2 = 0
\]

Solving for \( x \) gives:

\[
x = 1, x = 2
\]

Therefore, the general solution is:

\[
x_n = A(1)^n + B(2)^n
\]

To find the constants \( A \) and \( B \), we use the initial conditions:

\[
x_0 = 0, x_1 = 1
\]

Substituting these into the general solution:

\[
0 = A + B, 1 = A + 2B
\]

Solving this system of equations gives:

\[
A = -\frac{1}{2}, B = \frac{1}{2}
\]

Therefore, the explicit form of the sequence is:

\[
x_n = -\frac{1}{2}(1)^n + \frac{1}{2}(2)^n
\]

1. General Term

1.1 Problem 16.10. Given \( x_1 = 2, x_2 = 3, x_{2n} = x_{2n-1} + 2x_{2n-2}, x_{2n+1} = x_{2n} + x_{2n-1}, \) find an explicit expression for \( x_n \).

Solution: The recurrence can be written \( (a_{n+1} - 2a_n)^2 = 3a_n^2 - 2 \). Subtracting this equation from the one for \( n \rightarrow n+1 \), we get \((a_{n+2} - 4a_{n+1} + a_n)(a_{n+2} - a_n) = 0\). Since the sequence is strictly increasing, \( a_{n+2} - 4a_{n+1} + a_n = 0 \), and an induction finishes the proof.

1.2 Problem 16.11. Suppose that the sequence of integers \( (a_n) \) satisfies \( a_0 = 0 \), \( a_1 = a_2 = 1 \), \( a_{n+1} - 3a_n + a_{n-1} = 2 \cdot (-1)^n \). Prove that \( a_n \) is a perfect square.

Solution: The first terms of the sequence are 0, 1, 2, 4, 5, 8, 13. These numbers are the squares of the first terms of Fibonacci’s sequence, and one can prove by induction that \( a_n = f_n^2 \), where \( f_n \) is the \( n \)-th term of Fibonacci’s sequence. In the induction step it will be used the identity \( f_{n+2}^2 - f_{n+1}^2 = f_n f_{n+1} + (-1)^n = 0 \).

1.3 Problem 16.12. Let \( x_0, x_1 \) be fixed integers and let the sequence \( x_n \) be defined by

\[
x_{n+1} = \begin{cases} 
\frac{x_n}{2}, & x_n \text{ even} \\
x_n + x_{n-1}, & x_n \text{ odd}
\end{cases}
\]

Show there is \( N \) such that the sequence is periodic for \( n \geq N \).

Solution: Taking successively \( n = 1, 2, 3 \) and solving the corresponding equations we obtain \( a_1 = \frac{1}{2}, a_2 = -\frac{1}{3} \) and \( a_3 = \frac{1}{4} \). We prove inductively that...
\[ a_n = \frac{(-1)^{n+1}}{n+1}. \] Indeed, suppose that \( a_k = \frac{(-1)^{k+1}}{k+1} \) for \( k < n \). Using the identity \( \binom{k}{n+1} = \frac{1}{n+1} \binom{k}{n} \) the recurrence relation for \( n \) yields

\[
C_n a_k = \frac{n}{n+1} - \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{n+1} C_{n+1}^{k+1} = \\
\frac{1}{n+1} \left( n - \sum_{k=0}^{n+1} (-1)^k C_{n+1}^k + C_{n+1}^0 - C_{n+1}^1 + (-1)^{n+1} C_{n+1}^{n+1} \right) = \frac{(-1)^{n+1}}{n+1}
\]

**Problem 16.16.** Let \( a, b \) be fixed positive integers. Find the general solution to the recurrence relation \( x_0 = 0, x_{n+1} = x_n + a + \sqrt{b^2 + 4ax_n} \).

**Solution:** The recurrence relation can be arranged conveniently as

\[
ax_n + \frac{b^2}{4} + 2 \sqrt{ax_n + \frac{b^2}{4} + a^2} \quad \text{or} \quad \sqrt{ax_n + \frac{b^2}{4} + a}.
\]

Then \( \sqrt{ax_n + \frac{b^2}{4}} = \sqrt{ax_0 + \frac{b^2}{4}} + na, \) so \( x_n = n^2a + nb \).

**Problem 16.17.** Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence of numbers from the interval \((0,1]\) such that \( x_{2n} = 2x_n^2 - 1 \) for all natural numbers \( n \). Determine \( x_{2011} \).

**Solution:** For any \( n \in \mathbb{N} \), from \( x_{2n} = 2x_n^2 - 1 \) we obtain \( x_n > \frac{\sqrt{2}}{2} = \cos \frac{\pi}{2} \).

It is not difficult to prove by induction that \( x_n > \cos \frac{\pi}{2k} \) for all \( n \in \mathbb{N} \) and for all \( k \in \mathbb{N}, k \geq 2 \). Passing at the limit with \( k \to \infty \) in the inequalities

\[ 1 \geq x_n > \cos \frac{\pi}{2k}, \]

we obtain \( x_n = 1 \) for all \( n \in \mathbb{N} \).

**2. Periodicity**

**Problem 16.18.** Consider the sequence \( (x_n) \) defined by \( x_1 = a \) and \( x_{n+1} = \frac{x_n + \sqrt{2} - 1}{1 - x_n(\sqrt{2} - 1)} \), for any \( n \geq 1 \). Determine \( x_{2004} \).

**Problem 16.19.** Prove the sequence \( (x_n) \) satisfying \( x_{n+1} = \frac{2}{2-x_n} \) is periodic.

**Solution:** We observe that \( x_{n+1} = f(x_n) \) where \( f(x) = \frac{2}{2-x} \). Then \( x_{n+k} = f^k(x_n) \) for any \( n, k \in \mathbb{N} \), where

\[ f^k(x) = \underbrace{f \circ f \circ f \circ \ldots \circ f}_k(x) \]

But \( f^4(x) = x \), so \( x_{n+4} = x_n \), for any \( n \in \mathbb{N} \).
Problem 16.20. Let $k$ be a fixed positive integer and $(a_n)_n$ a sequence such that

$$a_{n+k} + a_{n-k} = a_n,$$

for all $n \geq k$.

Prove that $(a_n)_n$ is periodic.

Solution: We have $a_{n+2k} = a_{n+k} - a_n = -a_{n-k}$ and from here $a_{n+6k} = -a_{n+3k} = a_n$.

Problem 16.21. Determine the values of $x_0$ and $x_1$ for which the sequence which satisfies

$$x_{n+2} = x_{n+1} - x_n$$

in convergent.

Solution: We have $x_{n+3} = -x_n$ and $x_{n+6} = x_n$. But periodic sequences are convergent only if they are constant. Solving the $x_0 = x_1 = x_2 = x_1 - x_0$ we obtain $x_0 = x_1 = 0$.

Problem 16.22. Let $(x_n)_n$ be a sequence defined by $x_0 = a$ and $x_{n+1} = \frac{x_n - 1}{x_n + 1}$ for any integer $n \geq 0$. If $x_{2006} = 3$, find $a$.

Solution: Let $f(x) = \frac{x-1}{x+1}$. Then $x_{n+1} = f(x_n)$. Since $f(f(x)) = -\frac{1}{x}$ and $(f \circ f \circ f \circ f)(x) = x$, the sequence $(x_n)_n$ is periodic of period 4. Then $x_{2006} = x_2 = -\frac{1}{a}$, so $a = -\frac{1}{2006}$. 
CHAPTER 17

Convergence of sequences

1. Classic sequences

Problem 17.1. Prove that the sequence given by \( x_n = \left(1 + \frac{1}{n}\right)^n \) is increasing and bounded.

Problem 17.2. Using that \( e \) is the limit of the sequence \( \left(1 + \frac{1}{n}\right)^n \) prove that \( \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e. \)

Problem 17.3. Prove that the sequence defined by \( x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln(n + 1) \) is convergent.

Solution: Use the inequality

\[
\frac{1}{n+1} < \ln(n+1) - \ln n < \frac{1}{n}
\]

to prove \((x_n)\) is decreasing and bounded \(x_n \in (0, 1 - \ln 2)\).

The limit of this sequence is the Euler-Mascheroni constant.

Problem 17.4. Prove that the sequence defined by \( x_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n \) is convergent.

Problem 17.5. Evaluate the limit \( \lim_{n \to \infty} \sum_{k=1}^{n} \left( \frac{1}{2k-1} - \frac{1}{2k} \right) \).

Solution: We can use Riemann sums or the previous problem about the Euler-Mascheroni constant.

2. Convergence

Problem 17.6. Show that the sequences \( a_n = \sin n \) and \( b_n = \cos n \) diverge.

Solution: Suppose one of these two sequences, say \( a_n \), is convergent, and \( l = \lim a_n \). Then the equation

\[ \sin(n + 1) + \sin(n - 1) = 2 \sin n \cos 1 \]

proves that necessarily, \( l = 0. \) The sequence \( b_n \) must be also convergent to 0, since

\[ \sin(n + 1) - \sin(n - 1) = 2 \sin 1 \cos n \]

which contradicts \( a_n^2 + b_n^2 = 1. \)
Problem 17.7. Is there any sequence \( x_n \in (0, 1) \) such that \( x_n \) is divergent?

Solution: Yes. Take \( x_n = \frac{1}{\sqrt{2}} \) for \( n \) even and \( x_n = \frac{1}{\sqrt{3}} \) for \( n \) odd.

Problem 17.8. Is the sequence \( x_n = \sqrt{1 + \sqrt{2 + \sqrt{3 + \ldots + \sqrt{n}}} \} \) convergent?

Solution: Yes. The sequence is obviously monotonic and bounded as a consequence of exercise 9.46.

Problem 17.9. Prove that a sequence of real numbers \((a_n)\) which satisfies
\[
\frac{1}{8} (6a_n - a_{n-1}) \leq a_{n+1} \leq \frac{1}{6} (5a_n - a_{n-1})
\]
for all positive integers \( n \), has limit 0.

Problem 17.10. Consider the sequence \( a_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}} - 2\sqrt{n} \).

1. Prove that the sequence \((a_n)\) is convergent.
2. If \( a = \lim_{n \to \infty} a_n \), show that \( \frac{1}{2\sqrt{n+1}} < a_n - a < \frac{1}{2\sqrt{n}} \).
3. Evaluate \( \lim_{n \to \infty} \sqrt{n}(a_n - a) \).

Solution: a) Use the inequality \( \frac{1}{\sqrt{n+1}} < 2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} \) to show that \( a_n \) is decreasing and \( a_n \in (-2, 0) \).

b) The sequence \( b_n = a_n - \frac{1}{2\sqrt{n}} \) is increasing, hence is smaller than its limit \( a \).

The sequence \( c_n = a_n - \frac{1}{2\sqrt{n+1}} \) is decreasing, hence is greater than its limit \( a \).

c) As a consequence of b) the limit is \( \frac{1}{2} \).

Problem 17.11. Evaluate the limit \( \sin(\pi \sqrt{4n^2 + n} - 1) \).

Solution: Decarece \( 4n^2 + n - 1 = \left(2n + \frac{1}{4}\right)^2 - \frac{17}{16} \), the expression \( \sqrt{4n^2 + n} - 1 \) can be replaced with \( 2n + \frac{1}{4} \) for \( n \to \infty \). Then the limit can be written
\[
\lim_{n \to \infty} \sin \left(2n \pi + \frac{\pi}{4}\right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}.
\]

All this can be written also formally subtracting and adding \( 2n \) in the argument of \( \sin \) and then writing
\[
\sqrt{4n^2 + n} - 1 - 2n = \frac{n - 1}{\sqrt{4n^2 + n} - 1 + 2n} \to \frac{1}{4}
\]

Problem 17.12. Let \((x_n)\) be a sequence of real numbers. Then the following statements are equivalent:

(i) \( \lim_{n \to \infty} x_n = \infty \)
Solution : It is well known that (ii) and (iii) are consequences of (i). We prove that (ii) implies (i). Assume (ii) and also that \((x_n)\) doesn't converge to \(\infty\). Then \((2 + \sqrt{3})^n\) is increasing and since \(e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x\) we have \(f(x_n) \leq f(M) < e\). This contradicts the fact that \(\lim_{k \to \infty} f(x_k) = e\). Similarly we prove that (i) is a consequence of (iii).

Problem 17.13. Prove that the sequence \(a_n = \{(2 + \sqrt{3})^n\}\) is convergent and the sequence \(b_n = \{(1 + \sqrt{2})^n\}\) is divergent, where \(\{x\} = x - [x]\) is the fractional part of \(x\).

Solution : For any natural number \(n\) there exist integers \(A_n, B_n, C_n, D_n\) such that
\[
(2 + \sqrt{3})^n = A_n + B_n\sqrt{3}, \quad (2 - \sqrt{3})^n = A_n - B_n\sqrt{3}
\]
\[
(1 + \sqrt{2})^n = C_n + D_n\sqrt{2}, \quad (1 - \sqrt{2})^n = C_n - D_n\sqrt{2}
\]
Then \((2 + \sqrt{3})^n + (2 - \sqrt{3})^n = 2A_n \in \mathbb{Z}\). Since \((2 - \sqrt{3})^n \in (0, 1)\) it follows that \(a_n = \{(2 + \sqrt{3})^n\} = 1 - (2 - \sqrt{3})^n\) converges to 1.

Similarly, \((1 + \sqrt{2})^n + (1 - \sqrt{2})^n = 2C - n \in \mathbb{Z}\). For \(n\) odd we have \((1 - \sqrt{2})^n \in (-1, 0)\) so \(b_n = \{(1 + \sqrt{2})^n\} = -(1 - \sqrt{2})^n = (\sqrt{2} - 1)^n\) converges to 0 and for \(n\) even, \((1 - \sqrt{2})^n \in (0, 1)\), hence \(b_n = 1 - (1 - \sqrt{2})^n\) converges to 1.

Problem 17.14. Let \(k\) be a square-free positive integer and for each \(n \in \mathbb{N}\), \(a_n, b_n\) defined by \(a_n + b_n\sqrt{k} = (a + b\sqrt{k})^n\). Determine the limit \(\frac{a_n}{b_n}\).

Solution : We observe that \(a_n - b_n\sqrt{k} = (a - b\sqrt{k})^n\). Then
\[
a_n = \frac{(a + b\sqrt{k})^n + (a - b\sqrt{k})^n}{2}
\]
\[
b_n = \frac{(a + b\sqrt{k})^n - (a - b\sqrt{k})^n}{2\sqrt{k}}
\]
and
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \sqrt{k} \frac{1 + \left(\frac{a - b\sqrt{k}}{a + b\sqrt{k}}\right)^n}{1 - \left(\frac{a - b\sqrt{k}}{a + b\sqrt{k}}\right)^n} = \sqrt{k}
\]

Problem 17.15. Is the sequence \(x_n = \sqrt[1!]{\sqrt[2!]{\sqrt[3!]{\cdots}}/\sqrt{n!}}\) convergent?
Solution: The sequence \( \ln x_n = \sum_{k=2}^{n} \frac{\ln k!}{(k+1)!} \) is increasing. Moreover, \( \ln x_n \leq \sum_{k=2}^{n} \frac{k(k+1)!}{2(k+1)!} = \frac{e-1}{2} \), so \( \ln x_n \) is bounded. We used the inequality
\[
1 \cdot 2 \cdot \ldots \cdot k \leq \left( \frac{1+2+\ldots+k}{k} \right)^k.
\]

**Proposition 17.1.** Let \((a_n)\) be a sequence of positive numbers such that \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l < 1 \). Then the sequence \((a_n)\) is convergent toward 0.

**Proof:** Let \( \epsilon > 0 \) be such that \( l + \epsilon < 1 \). There is an \( N \) such that \( \frac{a_{n+1}}{a_n} < l + \epsilon \), for any \( n \geq N \). Then \( a_{N+k} < (l+\epsilon)^k a_N \), for any \( k \), and as a consequence \( \lim_{n \to \infty} a_n = 0 \).

**Applications** Evaluate the limit of the sequences \( \frac{n}{2^n}, \frac{2^n}{n!}, nq^n \), where \( q \in (-1, 1) \).

**Problem 17.16.** Evaluate the limit \( \lim_{n \to \infty} \left( 1 + \sqrt[n]{2n^2 + 1} \right)^{\frac{n}{2^n}} \).

**Solution:** The limit is the exponential of \( \lim_{n \to \infty} \sqrt[n]{2n^2 + 1} \). Take \( a_n = \frac{n^2 + 1}{e^{2n+1}} \), remark that \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^{2n+1} \frac{n^2}{e^{2n+1}} = 0 \), and use the previous problem.

**Problem 17.17.** Let \( r_n = \frac{p_n}{q_n} \) be a sequence of rational numbers convergent and such that \( q_n \) is positive for all \( n \).

(i) If \( r_n \) is convergent to a rational number \( r = \frac{p}{q} \), and \( r_n \neq r \) for any \( n \), then \( \lim_{n \to \infty} q_n = \infty \).

(ii) If \( r_n \) is convergent to an irrational number \( r \), then \( \lim_{n \to \infty} q_n = \infty \).

**Solution:** (i) Let \( \epsilon \) be a positive number. There is a positive integer \( N \) such that \( \left| \frac{p_n}{q_n} - \frac{p}{q} \right| = \frac{|p_n q - p q_n|}{q q_n} < \epsilon \) for \( n > N \). But \( |p_n q - p q_n| \neq 1 \), so \( \frac{1}{q q_n} \leq \frac{|p_n q - p q_n|}{q q_n} < \epsilon \). Hence \( \frac{1}{q q_n} \to 0 \).

(ii) Suppose that \( q_n \) is not convergent to \( \infty \). Then \( q_n \) has a bounded subsequence, and consequently a constant one \( (q_{n_k})_k \). Let \( m \) be the product of all distinct elements of this sequence. The sequence \( m r_{n_k} \) has only integer terms, so is convergent to an integer and the limit of \( (r_{n_k}) \) must be rational. Contradiction.

**Problem 17.18.** Evaluate \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \left( \left[ \frac{2n}{k} \right] - \left[ \frac{n}{k} \right] \right) \).
3. First order recurrences

Problem 17.19. Show that

a) The sequence \( a_n = \sum_{k=0}^{n} \cos k\theta \) is divergent.

b) The sequence \( b_n = \sum_{k=0}^{n} C_n^k \cos n\theta \) is either convergent to 0, or divergent.

Solution: a) It is known that \( a_n = \frac{\sin \frac{(2n+1)\theta}{2} + \sin \frac{\theta}{2}}{2\sin \frac{\theta}{2}} \), so \( a_n \) is divergent (see Exercise 17.6).

b) Consider \( c_n = \sum_{k=0}^{n} C_n^k \cos n\theta \). As consequence of Moivre’s equation

\[
    b_n + ic_n = (1 + \cos \theta + i \sin \theta)^n = \left(2 \cos \frac{\theta}{2}\right)^n \left(\cos \frac{n\theta}{2} + i \sin \frac{n\theta}{2}\right)
\]

so \( b_n = \left(2 \cos \frac{\theta}{2}\right)^2 \cos \frac{n\theta}{2} \). If \( 2 \cos \frac{\theta}{2} < 1 \), then \( b_n \) is convergent to 0. If \( 2 \cos \frac{\theta}{2} > 1 \), then \( b_n \) is divergent.

Problem 17.20. Let \( f : ]0, \infty[ \rightarrow \mathbb{R} \) be a continuous function and the sequence \( (x_n)_n \) defined by

\[
    f(1) + f(2) + \cdots - \int_{1}^{n+x_n} f(x)dx = C
\]

(\( C \) is Euler’s constant). Show that

i) \( x_n \in ]0,1[ \), \( \forall n \)

ii) \( x_n \rightarrow \frac{1}{2} \)

3. First order recurrences

Problem 17.21. Show that \( \frac{2}{\pi} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2} + \sqrt{2}}{2} \sqrt{2} + \frac{\sqrt{2} + \sqrt{2}}{2} \cdots \)

Solution: Consider the sequence \( a_n \) defined by \( a_1 = \frac{\sqrt{2}}{2} \) and \( a_{n+1} = \frac{\sqrt{2} + 2a_n}{2} \).

Then

\[
    \frac{\sqrt{2}}{2} + \frac{\sqrt{2} + \sqrt{2}}{2} \sqrt{2} + \frac{\sqrt{2} + \sqrt{2}}{2} \cdots = \lim_{n \rightarrow \infty} a_1 a_2 a_3 \cdots a_n
\]
One can prove by induction that 

$$a_n = \cos \frac{\pi}{2^{n+1}}.$$ 

So

$$a_1a_2 \cdots a_n = \cos \frac{\pi}{4} \cos \frac{\pi}{8} \cdots \cos \frac{\pi}{2^{n+1}} = \frac{\sin \frac{\pi}{2}}{2^n \sin \frac{\pi}{2^{n+1}}} \to \frac{2}{\pi} \text{ as } n \to \infty.$$

**Problem 17.22.** Evaluate

$$x = \sqrt[2207]{2207 - \frac{1}{2207} - \frac{1}{2207 - \cdots}}.$$

Express your answer in the form $x = \frac{a+b\sqrt{c}}{d}$, where $a, b, c, d$ are integers. [P1995]

**Solution:** We consider the sequence $(x_n)_n$ defined by $x_1 = 2207$ and $x_{n+1} = 2207 - \frac{1}{x_n}$. We prove by induction that $2206 < x_n \leq 2207$. For $n = 1$ the inequalities being satisfied, we suppose $2206 < x_k \leq 2207$ and one can easily see that $2206 < x_{k+1} = 2207 - \frac{1}{x_k} \leq 2207$. The function $f : (0, \infty) \to \mathbb{R}$, $f(x) = 2207 - \frac{1}{x}$ is increasing, and $x_2 < x_1$, therefore $x_{n+1} = f^{(n-1)}(x_2) < f^{(n-1)}(x_1) = x_n$. The sequence $(x_n)_n$ is convergent as decreasing and bounded, and $x = \lim x_n$.

Passing at the limit in the recurrence we obtain $x + \frac{1}{x} = 2207$. Successively, we have

$$\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = 47^2, \quad \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = 7^2, \quad \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = 3^2.$$ 

Hence $\sqrt{x}$ is the solution greater than 1 of the equation $\sqrt{x} + \frac{1}{\sqrt{x}} = 3$, thus $\sqrt{x} = \frac{3 + \sqrt{5}}{2}$.

**Problem 17.23.** Let $a \geq -6$ be a real number and the sequence $(a_n)_n$ defined by $a_0 = a$, and $a_{n+1} = \sqrt{6 + a_n}$. Prove the sequence is convergent and find its limit. If $k$ is a real number evaluate $\lim_{n \to \infty} n^k(a_n - l)$.

**Problem 17.24.** Let $a_1 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$ for $n \geq 1$. Prove the sequence is convergent and find its limit $l$. Evaluate $\lim_{n \to \infty} n^k(a_n - l)$.

**Problem 17.25.** The sequence $(a_n)_n$ is defined by $a_1 = 1$ and $a_{n+1} = \frac{1}{1 + a_n}$ for $n \geq 1$. Show that $(a_n)$ is convergent and find its limit.

**Solution:** The sequence is bounded, since $0 < a_n < 1$. The function $f : (0, 1) \to (0, 1)$ defined by $f(x) = \frac{1}{1 + x}$ is decreasing. Then $f \circ f$ is increasing, therefore the sequences $(a_{2n})$ and $(a_{2n+1})$ are monotonic. Let $x = \lim a_{2n}$ and $y = \lim a_{2n+1}$. Passing at the limit in the recurrence relation gives the system $x = \frac{1}{1 + y}$ and $y = \frac{1}{1 + x}$ with the only convenient solution $x = y = \frac{\sqrt{5} - 1}{2}$.

**Problem 17.26.** Let the sequence $(x_n)_n$ be defined by $x_0 \in \mathbb{R}$ and $x_{n+1} = \cos x_n$. Show the sequence is convergent to the unique solution of the equation $\cos x = x$. 

Let $x_0 \in \mathbb{R}$ and $x_{n+1} = \frac{1}{1 + x_n}$ for $n \geq 1$. Show that $(a_n)$ is convergent and find its limit.
Solution: One can prove that $x_n \in (0, 1]$, for any $n \geq 2$. Indeed, $x_1 = \cos x_0 \in [-1, 1] \subset [-\pi/2, \pi/2]$, so $x_2 \in [0, 1) \subset (0, \pi/2)$ and we continue by induction. The function $f(x) = \cos x$ is decreasing on the interval $(0, 1]$, so $f \circ f$ is increasing. Then the subsequences $(x_{2n})$ and $(x_{2n+1})$ are monotonic. Let $a = \lim x_{2n}$ and $b = \lim x_{2n+1}$. Passing at the limit in the equations $x_{2n+1} = \cos x_{2n}$ and $x_{2n} = \cos x_{2n-1}$ gives $b = \cos a$ and $a = \cos b$. Then $\cos \cos a = a$ and $\cos \cos b = b$. Consider the function $g(x) = x - \cos \cos x$. Its derivative $g'(x) = 1 - \sin x \sin \cos x$ is strictly positive, so $g$ is increasing. Since $g(0) = -\cos 1 < 0$ and $g(1) = 1 - \cos \cos 1 > 0$, the equation $g(x) = 0$ has an unique solution, solution which lies in the interval $(0, 1)$. Then $a = b$ and the sequence $(x_n)$ is convergent. If $l$ is the unique solution of the equation $\cos l = l$, then $g(l) = 0$, so $\lim x_n = l$.

Problem 17.27. Study the convergence of the sequence defined by $x_1 = \sqrt{2}$, and $x_{n+1} = (\sqrt{2})^{x_n}$.

Solution: By induction one shows the sequence is increasing and bounded above by 2. The limit $x$ of the sequence satisfies $x = (\sqrt{2})^x$, equation which is equivalent with $f(x) = \ln x - x \ln 2 = 0$. But the function $f$ has the only root $x = 2$ on the interval $(\sqrt{2}, \infty)$.

Problem 17.28. Study the convergence of the sequence $x_n$ defined by $x_1 = x > 0$, $x_{n+1} = x + \frac{1}{x_n}$.

Solution: By induction $x_{2n-2} > x_{2n} > x_{2n+1} > x_{2n-1}$ for all $n > 1$. Then there are well defined $u = \lim x_{2n}$ and $v = \lim x_{2n+1}$, the limit in the recurrence we get $u = x + \frac{1}{u}$ and $v = x + \frac{1}{v}$. We substract, and since $uv \neq 1$ (otherwise $x = 0$), we get $u = v$. The sequence is then convergent to the positive solution of $u^2 - xu - 1 = 0$.

Problem 17.29. Let $x_0$ be fixed, and $a, b$ satisfy $\sqrt{b} < a$. Define $x_n$ recursively by $x_{n+1} = \frac{ax_n + b}{x_n + a}$, $n = 1, 2, 3, \ldots$ Prove that $(x_n)_n$ is convergent and evaluate the limit.

Solution: The function $f(x) = \frac{ax + b}{x + a} = a - \frac{a^2 - b}{x + a}$ is increasing for positive values of $x$. Using $x_{n+1} = f(x_n)$, one can easy prove that $0 < x_n < a$. Also this shows that the sequence is monotonic, increasing if $x_0 \leq x_1$ and decreasing if $x_0 \geq x_1$. The sequence is therefore convergent. Let $l = \lim_{n \to \infty} x_n$. Passing at the limit in the recurrence gives $l = \frac{al + b}{l + a}$ with the only positive solution $l = \sqrt{b}$.

Problem 17.30. Let $(x_n)$ be a sequence defined by $x_0 \in (0, 1)$ and $x_{n+1} = x_n - x_n^2 + x_n^3 - x_n^4$. Prove that the sequence is convergent and evaluate the limit.

Solution: The sequence is decreasing, since $x_{n+1} - x_n = -x_n^2(1-x_n+x_n^2) \leq 0$. We prove by induction that $x_n > 0$. Suppose $x_n > 0$. Then $x_{n+1} = x_n(1-x_n)(1+x_n^2) > 0$, for $x_n \leq x_0 < 1$. The sequence $(x_n)$ is convergent as a decreasing and bounded below sequence. Let $l = \lim x_n$. Passing at the limit in the recurrence which defines the sequence, we get $l = l - l^2 + l^3 - l^4$, therefore $l = 0$. 

3. First Order Recurrences
Problem 17.31. Let the sequence \( (x_n) \) be defined by \( x_1 = 1 \) and \( x_{n+1} = 1 + \frac{1}{1 + x_n} \), for any \( n \geq 1 \). Show that \( (x_n) \) is convergent and \( \lim x_n = \sqrt{2} \).

**Solution:** One can prove by induction that \( x_n \in [1, 2) \), for any \( n \geq 1 \). The function \( f(x) = 1 + \frac{1}{1 + x} \) is decreasing on the interval \((1, 2]\), so \( f \circ f \) is increasing. Therefore the subsequences \( (x_{2n}) \) and \( (x_{2n+1}) \) are monotonic. Let \( a = \lim x_{2n} \) and \( b = \lim x_{2n+1} \). Passing at the limit in the equations \( x_{2n+1} = f(x_{2n}) \) and \( x_{2n} = f(x_{2n-1}) \) gives \( a = f(b) \) and \( b = f(a) \). Then \( a = f(f(a)) \) and \( b = f(f(b)) \). But the function \( f \circ f \) is strictly increasing, so \( a = b \). Solving the equation \( a = f(a) \), gives the only convenient solution \( a = \sqrt{2} \).

Problem 17.32. Prove that \( \sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \ldots}} \).

**Solution:** Let \( x_1 = 1 \) and \( x_n = 1 + \frac{1}{2 + \frac{1}{2 + \ldots}} \), for \( n \geq 2 \), where 2 appears \( n-1 \) times. Then the sequence \( (x_n) \) satisfies the recurrence \( x_{n+1} = 1 + \frac{1}{1 + x_n} \). We can continue like in the previous problem.

Problem 17.33. A sequence \( (a_n) \) is given by \( a_1 = \sqrt{2} \) and \( a_{n+1} = \sqrt{2 + a_n} \). Show that \( (a_n) \) is convergent and find its limit.

**Solution:** One can prove by induction that \( a_n = 2 \cos \frac{\pi}{2^{n+1}} \).

Problem 17.34. Show that the sequence defined by \( a_1 = 1 \) and \( a_{n+1} = 3 - \frac{1}{a_n} \) is convergent and find its limit. In general, study for what values of \( a_1 \) is the sequence \( a_n \) convergent.

**Solution:** The function \( f(x) = 3 - \frac{1}{x} \) is increasing, therefore \( (a_n) \) is monotonic. Moreover, \( 1 \leq a_n < 3 \), so \( (a_n) \) is convergent. The limit is \( \frac{\sqrt{5} + 1}{2} \).

Problem 17.35. Show that the sequence defined by \( a_1 = 2 \), \( a_{n+1} = \frac{1}{3 - a_n} \) satisfies \( 0 < a_n \leq 2 \) and is decreasing to a limit to find.

**Solution:** Consider the function \( f(x) = \frac{1}{3 - x} \) which is increasing on \((0, 1] \) and satisfies \( f((0, 1]) \subset (0, 1] \). Since \( a_2 = 1 < a_1 \) and \( a_{n+1} = f(a_n) \), the sequence is decreasing and \( a_n \in (0, 1] \), for \( n \geq 2 \). Then the sequence is convergent and the limit must be a fix point of \( f \). There are two fixed points and the only one in the interval \((0, 1] \) is \( \frac{3 - \sqrt{5}}{2} \).
4. Second order recurrences

**Problem 17.36.** A sequence is defined by the equation \( a_n = \frac{a_{n-1} + a_{n-2}}{2} \). Show that \((a_n)\) is convergent and find the limit.

**Solution:** The recurrence writes \( a_n - a_{n-1} = -\frac{1}{2}(a_{n-1} - a_{n-2}) \). Then \( a_n - a_{n-1} = \left( -\frac{1}{2} \right)^{n-2} (a_2 - a_1) \), and adding these relations \( a_n = a_1 + \sum_{k=0}^{n-2} \left( -\frac{1}{2} \right)^k (a_2 - a_1) = \frac{2}{3} (1 - (-1/2)^{n-1}) (a_2 - a_1) \). The limit of \( a_n \) is then \( \frac{2a_2 + a_1}{3} \).

**Problem 17.37.** The sequence \((x_n)\) is defined by the recurrence \( x_0 = 0, x_1 = 1, x_{n+1} = \frac{x_n + nx_{n-1}}{n+1}, n > 1 \). Determine \( L = \lim_{n \to \infty} x_n \).

**Solution:** The recurrence can by written conveniently
\[
(n + 1)(x_{n+1} - x_n) + n(x_n - x_{n-1}) = 0.
\]
Therefore \( n(x_n - x_{n+1}) = (-1)^{n+1} \) and the recurrence relation becomes \( x_n - x_{n-1} = (-1)^{n+1} \). We obtain \( x_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \). In particular
\[
x_{2n} = \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} \right) - 2 \left( \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2n} \right)
= \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{2n} - \ln(2n) \right) - \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n \right) + \ln 2
= \left( 1 + \frac{1}{2} + \ldots + \frac{1}{2n+1} \right) - \ln(2n+1)
= \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n \right) + \ln \frac{2n+1}{n},
\]
and we see now easily that \( L = \ln 2 \).

**Problem 17.38.** Let the sequences \((a_n)\) and \((b_n)\) be defined by \( a_1 = a \geq 0 \), \( b_1 = 1 \) and \( a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{b_n a_{n+1}} \). Show that \((a_n)\) and \((b_n)\) are convergent and find their limit.

**Solution:** There is \( t \in [0, \pi/2] \) such that \( a = \cos t \). Then by induction one can prove that
\[
b_n = \cos \frac{t}{2} \cos \frac{t}{2^2} \ldots \cos \frac{t}{2^{n-1}}, a_n = b_n \cos \frac{t}{2^{n-1}}.
\]
Since \( b_n = \frac{\sin t}{2^{n-1} \sin \frac{t}{2^{n-1}}} \), the sequences are convergent and \( \lim a_n = \lim b_n = \frac{\sin t}{t} \), with the convention \( \frac{\sin 0}{0} = 1 \).

**Problem 17.39.** Let \( a_0, b_0 \) be positive numbers and the sequences \((a_n)\) and \((b_n)\) be defined by \( a_{n+1} = \frac{a_n + b_n}{2} \) and \( b_{n+1} = \sqrt{a_n b_n} \). Show that the sequences \((a_n)\) and \((b_n)\) are convergent and have the same limit.
Solution: One can prove that $a_n > a_{n+1} > b_{n+1} > b_n > 0$, for any $n \geq 1$. Indeed, using the inequality between the arithmetic and geometric means, gives $a_n > b_n$ for any $n \geq 1$ and this implies the other two inequalities. Then the two sequences are monotonic and bounded, so they are convergent. Let $a = \lim a_n$ and $b = \lim b_n$.

Passing at the limit in the equation $a_{n+1} = \frac{a_n + b_n}{2}$, gives $a = b$.

**Problem 17.40.** Let $(x_1, y_1) = (0.8, 0.6)$ and let $x_{n+1} = x_n \cos y_n - y_n \sin y_n$ and $y_{n+1} = x_n \sin y_n + y_n \cos y_n$ for $n = 1, 2, 3, \ldots$. For each of $\lim_{n \to \infty} x_n$ and $\lim_{n \to \infty} y_n$, prove that the limit exists and find it or prove that the limit does not exist. [P1987]

Solution: Let $y_0 = \arcsin \frac{3}{5}$. Then $x_1 = \cos y_0, y_1 = \sin y_0$ and by induction

\begin{align*}
x_n &= \cos(y_0 + y_1 + \ldots + y_{n-1}) \\
y_n &= \sin(y_0 + y_1 + \ldots + y_{n-1})
\end{align*}

Define the sequence $(s_n)_n$ by $s_n = y_0 + y_1 + \ldots + y_n$. This sequence satisfies the recurrence $s_n = s_{n-1} + \sin s_{n-1} = f(s_{n-1})$, where $f(x) = x + \sin x$. The function $f$ satisfies $f(0) = 0, f(\pi) = \pi, f'(x) = 1 + \cos x \geq 0$ so it is an increasing bijection form $[0, \pi]$ to $[0, \pi]$. But $s_0 = \arcsin \frac{3}{5} \in [0, \pi]$ and $s_n = f^{(n)}(s_0)$, so the sequence $(s_n)_n$ is increasing and has $\pi$ as upper bound. Let $s = \lim s_n$. Then $s_0 \leq s \leq \pi$ and $\sin s = 0$, so $s = \pi$. But the equations 4.1 can be written $x_n = \cos s_{n-1}$ and $y_n = \sin s_{n-1}$ so $\lim x_n = -1$ and $\lim y_n = 0$.

5. Subsequences

**Problem 17.41.** Prove that $(x_n)$ is convergent if and only if the subsequences $(x_{2n}), (x_{2n+1}), (x_{3n})$ are convergent.

Solution: Suppose these three subsequences are convergent and let $\lim x_{2n} = a$, $\lim x_{2n+1} = b$ and $\lim x_{3n} = c$. Since $(x_{6n})$ is a subsequence of both $(x_{2n})$ and $(x_{3n})$ we have $a = c$. Also, $(x_{6n+3})$ is a subsequence of both $(x_{2n+1})$ and $(x_{3n})$, so $b = c$. Since the subsequences $(x_{2n})$ and $(x_{2n+1})$ cover the sequence $(x_n)$ and are convergent to the same limit $a = b$, the sequence $(x_n)$ is convergent.

6. The Cesaro Stolz theorem

**Proposition 17.2.** (Cesaro-Stolz) Let $(a_n)_n$ and $(b_n)_n$ be sequences of real numbers such that $b_n$ is strictly increasing to $\infty$ and the limit $l = \lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$ exists in $[-\infty, \infty]$. Then $\lim_{n \to \infty} \frac{a_n}{b_n} = l$.

**Problem 17.42.** Evaluate the limit

**Problem 17.43.** Evaluate the limit

**Problem 17.44.** Evaluate the limit $\lim_{n \to \infty} \frac{1! + 2! + \ldots + n!}{n!}$

**Problem 17.45.** Evaluate the limit $\lim_{n \to \infty} \frac{1! + 2! + \ldots + n!}{(2n)!}$
Problem 17.46. Evaluate the limit \( \lim_{n \to \infty} \frac{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}}{\sqrt{n}} \).

Problem 17.47. Evaluate the limit \( \lim_{n \to \infty} \frac{1}{n} \left( 1 + \frac{1}{1 + \sqrt{2}} + \cdots + \frac{n}{1 + \sqrt{2} + \cdots + \sqrt{n}} \right) \).

Problem 17.48. Evaluate the limit \( \lim_{n \to \infty} \frac{1 + \sqrt{2} + \cdots + \sqrt{n}}{n\sqrt{n}} \).

Problem 17.49. Evaluate the limit \( \lim_{n \to \infty} \frac{1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}}{n} \), a fixed real number.

Problem 17.50. Evaluate the limit \( \lim_{n \to \infty} \frac{a^{\frac{1}{n}} + \cdots + a^{\frac{1}{n^m}}}{n} \), a fixed real number.

Problem 17.51. Evaluate the limit \( \lim_{n \to \infty} \frac{1 + 2^2 \sqrt{2} + 3^2 \sqrt{3} + \cdots + n^2 \sqrt{n}}{n(n+1)(n+2)} \).

Problem 17.52. Evaluate the limit \( \lim_{n \to \infty} \frac{1}{n^2} \sum_{k=1}^{n} \ln(1 + kn) \).

Problem 17.53. Evaluate the limit \( \lim_{n \to \infty} \frac{\ln(n!)}{n^m}, m \) fixed real number.

Problem 17.54. Evaluate the limit \( \lim_{n \to \infty} \frac{1^m + 2^m + \cdots + n^m}{n^m} - \frac{n}{m + 1}, m \) fixed real number.

Problem 17.55. Evaluate the limit \( \lim_{n \to \infty} \frac{\sum_{k=m}^{n} (2k+1)^2 - an^p}{n^q} \), \( p, q, m \in \mathbb{N} \) and \( a \in \mathbb{R} \).

Problem 17.56. Evaluate the limit \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{a\sqrt{k} + b}{c\sqrt{k} + d} \).

Problem 17.57. Evaluate the limit \( \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \frac{1}{a_k}}{\ln a_n} \), where \( a_n = a_1 + (n-1)r \) for any \( n \geq 1 \).

Problem 17.58. Evaluate the limit \( \lim_{n \to \infty} \frac{\sum_{k=1}^{n} \frac{k^p}{n^{p+1}}} = \frac{1}{p + 1} \).

Problem 17.59. Evaluate the limit \( \lim_{n \to \infty} \frac{\cos \frac{\pi}{n} + \cos \frac{\pi}{n+1} + \cdots + \cos \frac{\pi}{2n}}{n} \).

Problem 17.60. Evaluate the limit \( L = \lim_{n \to \infty} \frac{n}{2^n} \sum_{k=1}^{n} \frac{2^k}{k} \).
Solution: We use the Cesaro-Stolz lemma for \( a_n = \sum_{k=1}^{n} \frac{2^k}{k} \) and \( b_n = \frac{2^n}{n} \). Then

\[
L = \lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \to \infty} \frac{\frac{2^{n+1}}{n+1} - \frac{2^n}{n}}{1 - \frac{n+1}{2n}} = 2
\]

PROBLEM 17.61. Let \((a_n)\) be a sequence such that \( \lim n(a_{n+1} - a_n) = 1 \). Prove that \( \lim a_n = \infty \) and \( \lim \sqrt{a_n} = 1 \).

Solution: Let \( b_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \). The hypothesis writes \( \lim a_n = 1 \). Using the Cesaro-Stolz theorem \( \lim \frac{a_n}{b_n} = 1 \). As an immediate consequence \( \lim a_n = \infty \).

On the other side, \( \lim \frac{a_{n+1}}{a_n} = \lim \frac{a_{n+1}b_{n+1}b_n}{b_{n+1}b_n a_n} = 1 \). Then using the part (ii) of the problem 17.63, \( \lim \sqrt{a_n} = 1 \).

PROBLEM 17.62. Let \( a_n \) be a sequence. Consider the following statements:

(p1) the limit \( \lim n(a_{n+1} - a_n) \) exists and is finite
(p2) the limit \( \lim \frac{a_1 + a_2 + \ldots + a_n}{n} \) exists and is finite
(p3) the sequence \( a_n \) is convergent.

Prove that:

(i) (p1) doesn’t imply (p3)
(ii) (p2) doesn’t imply (p3)
(iii) (p1) and (p2) imply (p3).

Solution: (i) Counter-example \( a_n = \sum_{k=1}^{n} \frac{1}{k} \).

(ii) Counter-example \( a_n = (-1)^n \).

(iii) Let \( b_n = n(a_{n+1} - a_n) \), and \( c_n = \frac{1}{n} \sum_{k=1}^{n} b_k \). Since \( b_n \) is convergent, then \( c_n \) is also convergent. Therefore \( a_{n+1} = c_n + \frac{1}{n} \sum_{k=1}^{n} a_k \) is also convergent.

PROBLEM 17.63. (i) Let \( (x_n) \) be a convergent sequence of positive real numbers. Then \( \lim \frac{x_1 + x_2 + \ldots + x_n}{n} = \lim \sqrt{x_1x_2\ldots x_n} = \lim x_n \).

(ii) If \( (a_n) \) is a convergent sequence of positive real numbers such that the sequence \( \frac{a_{n+1}}{a_n} \) is convergent, then the sequence \( \sqrt{a_n} \) is convergent and \( \lim \sqrt{a_n} = \lim \frac{a_{n+1}}{a_n} \).

Solution: (i) The equality \( \lim \frac{x_1 + x_2 + \ldots + x_n}{n} = \lim x_n \) is an immediate consequence of the Cesaro-Stolz theorem. For the last part remark \( \sqrt{x_1x_2\ldots x_n} = e^\frac{\ln x_1 + \ln x_2 + \ldots + \ln x_n}{n} \) and use the first part.
(ii) Use (i) for \( x_n = \frac{a_{n+1}}{a_n} \), or use the problem 17.64.

7. The Cauchy - d’Alembert theorem and applications

**Proposition 17.3. (Cauchy-d’Alembert)** Let \( (a_n)_n \) be a sequence of positive real numbers. If \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = l \in [0, \infty] \), then \( \lim_{n \to \infty} \sqrt[n]{a_n} = l \).

**Problem 17.64.** Let \( (x_n)_n \) be a sequence of positive terms. Then

\[
\lim \frac{x_{n+1}}{x_n} \leq \lim \sqrt[n]{x_n} \leq \lim_{n \to \infty} \sqrt[n]{x_n} \leq \lim \frac{x_{n+1}}{x_n}
\]

**Solution:** The inequality in the middle is obvious, we prove the other two inequalities. Let \( l = \lim_{n \to \infty} \frac{x_{n+1}}{x_n} \). If \( l = 0 \), then obviously the left most inequality is satisfied. If \( l > c > 0 \), there is \( N \geq 1 \), such that \( \frac{x_{n+1}}{x_n} \geq c \), for \( n \geq N \). Inductively one shows that \( x_n \geq c^{n-N} x_N \) for \( n > N \). Then \( \lim \sqrt[n]{x_n} \geq \lim_{n \to \infty} \sqrt[n]{c^{n-N} x_N} = c \). Therefore, \( \lim \sqrt[n]{x_n} \geq l \).

Let \( L = \lim_{n \to \infty} \frac{x_{n+1}}{x_n} \). Since for \( L = \infty \), the right most inequality is satisfied, we can suppose \( L \leq c \), where \( c \) is finite. There is \( N \) such that \( \frac{x_{n+1}}{x_n} \leq c \), for \( n \geq N \). By induction \( x_n \leq c^{n-N} x_N \) for any \( n \geq N \), or \( \sqrt[n]{x_n} \leq c^{n-N} \sqrt[n]{x_N} \). Passing at the limit we get \( \lim \sqrt[n]{x_n} \leq c \) and consequently \( \lim \sqrt[n]{x_n} \leq L \).

**Problem 17.65.** Evaluate the limit \( \lim_{n \to \infty} \sqrt[n]{\tan \frac{n\pi}{2n+1}} \).

**Problem 17.66.** Evaluate the limit \( \lim_{n \to \infty} \sqrt[n]{P(n)} \), where \( P \) is a polynomial with real positive coefficients.

**Problem 17.67.** Evaluate the limit \( \lim_{n \to \infty} \sqrt[n]{\sum_{k=1}^{n} \frac{1}{k}} \).

**Problem 17.68.** Evaluate the limit \( \lim_{n \to \infty} \frac{\sqrt[n]{n! \cdot 2^n}}{n} \).

**Problem 17.69.** Evaluate the limit \( \lim_{n \to \infty} \sqrt[n]{\frac{(n!)^2(n + a)}{n^k}} \), where \( a, k \) are fixed real numbers.

**Problem 17.70.** Prove that \( \lim_{n \to \infty} \left( n^{1/n} + (n + 1)! - \sqrt[n]{n!} \right) = e^{-1} \).

**Solution:** Denote \( a_n = \frac{n^n}{n!} \). Since \( \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim \left( 1 + \frac{1}{n} \right)^n = e \), we have \( \lim \sqrt[n]{a_n} = \lim \frac{n}{\sqrt[n]{n!}} = e \). As a consequence the sequence \( c_n = \frac{n^{1/n} + (n + 1)!}{\sqrt[n]{n!}} \) has the limit 1. On the other side \( \lim (c_n)^n = \lim \frac{n + 1}{\sqrt[n]{(n + 1)!}} = e \). Then taking \( b_n = \frac{n^{1/n} + (n + 1)!}{\sqrt[n]{n!}} \), from \((c_n)^n = (1 + (c_n - 1))^{1/(c_n - 1)}\) we have \( \lim b_n = 1/e \).
8. Squeeze theorem

**Proposition 17.4. (The squeeze principle)** Let \((x_n)_n, (y_n)_n, (z_n)_n\) be sequences of real numbers such that \(x_n \leq y_n \leq z_n\) for any \(n\), and \((x_n)_n, (z_n)_n\) are convergent with \(\lim_{n \to \infty} x_n = \lim_{n \to \infty} z_n = l\). Then \((y_n)_n\) is convergent and \(\lim_{n \to \infty} y_n = l\).

**Problem 17.71.** Evaluate the limit of the sequence \(a_n = \sum_{k=1}^{n} \frac{k^2 + k}{n^3 + k}\).

**Solution:** We have the inequalities \(\sum_{k=1}^{n} \frac{k^2 + k}{n^3 + n} \leq a_n \leq \sum_{k=1}^{n} \frac{k^2 + k}{n^3 + 1}\), or

\[
\frac{n(n+1)(2n+1)}{6} \frac{n(n+1)}{2} \leq a_n \leq \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}.
\]

so \(\lim a_n = \frac{1}{3}\).

**Problem 17.72.** Evaluate the limit of the sequence defined by \(a_n = \sum_{k=1}^{n} \frac{n + k}{n^3 + k}\).

**Problem 17.73.** Evaluate the limit of the sequence defined by \(a_n = \prod_{k=1}^{n} \left(1 + \frac{1}{\sqrt{n^2 + k}}\right)\).

**Problem 17.74.** Consider the sequence \(x_n = \left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{2n}\right)\left(1 + \frac{1}{3n}\right)\ldots \left(1 + \frac{1}{n^2}\right)\), for any \(n \in \mathbb{N}\). Prove that the sequence is bounded above by 2 and it is convergent to 1.

**Solution:** We use \(x - \frac{x^3}{3} < \ln(1 + x) < x\) and we get

\[
\ln x_n = \sum_{k=1}^{n} \ln \left(1 + \frac{1}{kn}\right) < \sum_{k=1}^{n} \frac{1}{kn} < \frac{\ln(1 + n)}{n} \leq \ln 2.
\]

For the convergence we use both inequalities above and the squeeze theorem.

**Solution:** We use the inequalities \(x - \frac{x^2}{2} < \ln(1 + x) < x\) for all \(x > 0\). Then

\[
\sum_{k=1}^{n} \left(\frac{1}{\sqrt{n^2 + k}} - \frac{1}{2(n^2 + k)}\right) \leq \ln a_n \leq \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k}}
\]

The left and right sequences have the limit 1, so the sequence in the center has also limit 1. Hence the limit that we are looking for is \(e\).

**Problem 17.75.** Evaluate the limit of the sequence \(x_n = \left(\frac{1 \cdot 3 \cdot \ldots \cdot (2n - 1)}{2 \cdot 4 \cdot \ldots \cdot (2n)}\right)^{a}\), where \(a\) is a fixed real number.

**Solution:** For \(0 < x < y\) we have \(\frac{x}{y} < \frac{x+1}{y+1}\) which leads to

\[
\frac{1 \cdot 2 \cdot 4 \cdot \ldots \cdot (2n - 2)}{2 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 1)} < \frac{1 \cdot 3 \cdot \ldots \cdot (2n - 1)}{2 \cdot 4 \cdot \ldots \cdot (2n)} < \frac{2 \cdot 4 \cdot \ldots \cdot (2n)}{3 \cdot 5 \cdot \ldots \cdot (2n + 1)}
\]
We obtain
\[
\frac{1}{2\sqrt{n}} < \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot \ldots \cdot (2n)} < \frac{1}{\sqrt{2n+1}}
\]
The limit is then 1 for \(\alpha = 0\), 0 for \(\alpha > 0\), and \(\infty\) for \(\alpha < 0\).

**Problem 17.76.** Evaluate the limit \(\lim_{n \to \infty} \sqrt[n]{c_1 a_1^n + c_2 a_2^n + \ldots + c_n a_n^n}\), where \(c_1, c_2, \ldots, c_n, a_1, a_2, \ldots, a_n\) are fixed positive numbers.

**Problem 17.77.** Evaluate the limit of the sequence defined by \(a_n = \frac{[x] + [2x] + [3x] + \ldots + [nx]}{n^2}\).

**Problem 17.78.** Evaluate the limit of the sequence defined by \(a_n = \sum_{k=1}^{n} \frac{\sin k}{n^2 + k}\).

**Problem 17.79.** Evaluate the limit of the sequence defined by \(a_n = \sum_{k=1}^{n} \left( \sqrt{1 + \frac{k^{p-1}}{n^p}} - 1 \right)\), where \(p\) is a fixed integer \(\geq 2\).

**Problem 17.80.** Evaluate the limit of the sequence defined by \(a_n = \sum_{k=1}^{n} \frac{\sin^p \frac{\pi}{n+k}}{n+k}\), where \(p\) is a fixed real number.

**9. Limsup and liminf**

**Problem 17.81.** Let \((x_n)\) be a sequence of strictly positive real numbers. Prove that
\[
\limsup \left( \frac{x_{n+1} + x_1}{x_n} \right)^n \geq e
\]

**Solution:** We prove that for any \(k \in \mathbb{N}\) there is \(n_k \in \mathbb{N}\) such that \(\frac{x_{n_k+1} + x_1}{x_{n_k}} \geq 1 + \frac{1}{n_k}\). Indeed if we assume the contrary, there exist \(k \in \mathbb{N}\) such that for any \(n \geq k\) we have \(\frac{x_{n+1} + x_1}{x_n} < 1 + \frac{1}{n}\) which can be rewritten \(\frac{x_n}{n} - \frac{x_{n+1}}{n+1} > \frac{x_1}{n+1}\). We sum the inequalities obtained for \(n = k, k+1, k+2, \ldots, k+p\) and obtain
\[
\frac{x_k}{k} - \frac{x_{k+p}}{k+p} > x_1 \left( \frac{1}{k} + \frac{1}{k+1} + \ldots + \frac{1}{k+p} \right)
\]
The left hand side is less than \(\frac{x_k}{k}\) while the right hand side converges to \(\infty\). Contradiction.
CHAPTER 18

Series

1. Convergence

Problem 18.1. a) Determine the sequence \( x_n \) such that 
\[
\sum_{k=1}^{n} x_k = \frac{n(n+1)(4n+5)}{6}.
\]
b) Determine the real numbers \( a, b \) such that 
\[
\frac{1}{x_n} = \frac{a}{n} + \frac{b}{2n+1}, \text{ for any } n.
\]
c) Evaluate the sum of the series 
\[
\sum_{n=1}^{\infty} \frac{1}{x_n}.
\]

Problem 18.2. Let \( (x_n) \) be a sequence of real numbers defined by 
\( x_1 \in (0, 1) \) and 
\( x_{n+1} = x_n - nx_n^2 \) for any \( n \geq 1 \). Prove that the series \( \sum_{n=1}^{\infty} x_n \) is convergent.

Solution: We prove by induction that \( x_n \in \left[0, \frac{1}{n\sqrt{n}}\right) \). For \( n = 1 \) the statement follows from the hypothesis \( x_1 \in (0, 1) \). Assume the statement is true for an \( n \). The function \( f(x) = x - nx^2 \) is increasing on the interval \( \left[0, \frac{1}{2n}\right] \) and \( \left[0, \frac{1}{n\sqrt{n}}\right] \) \( \subset \) 
\[
\left[0, \frac{1}{2n}\right]
\]
for \( n \geq 4 \). Then \( 0 = f(0) < f(x_n) = x_{n+1} < f\left(\frac{1}{n\sqrt{n}}\right) = \frac{\sqrt{n} - 1}{n^2} = \frac{n^2 - 1}{n^2} \cdot \frac{1}{n+1} \cdot \frac{1}{\sqrt{n} + 1} < \frac{1}{(n+1)^{\sqrt{n}+1}}.
\]

Problem 18.3. Let \( a_k \) be a sequence of positive numbers. Define \( s_n = \sum_{k=1}^{n} a_k \).

Prove that

(i) \( \sum_{n=1}^{\infty} \frac{a_n}{s_n^2} \) converges. (In fact, \( \sum_{n=1}^{\infty} \frac{a_n}{s_n^\alpha} \) converges for all \( \alpha > 1 \).)

(ii) If \( (s_n) \) is divergent then \( \sum_{n=1}^{\infty} \frac{a_n}{s_n} \) is divergent.

Solution: (i) From 
\[
\frac{a_n}{s_n^2} = \frac{s_n - s_{n-1}}{s_n^2} \leq \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}
\]
we have 
\[
\sum_{n=1}^{\infty} \frac{a_n}{s_n^2} \leq \frac{a_1}{s_1^2} + \sum_{n=2}^{\infty} \left( \frac{1}{s_{n-1}} - \frac{1}{s_n} \right) < \frac{2}{a_1}.
\]
(ii) Denote \( \sigma_n = \sum_{k=1}^{n} \frac{a_k}{s_k} \). Since the sequence \( (s_n) \) is increasing, we have
\[
|\sigma_{n+p} - \sigma_n| \geq \frac{a_{n+1} + a_{n+2} + \ldots + a_{n+p}}{s_{n+1}} = 1 - \frac{s_n}{s_{n+p}}
\]
for any \( n \) and \( p \). Since \( (s_{n+p})_p \) is divergent there is \( p_0 \) such that \( \frac{s_n}{s_{n+p}} \leq \frac{1}{2} \) for \( p > p_0 \), hence \( |\sigma_{n+p} - \sigma_n| \geq \frac{1}{2} \) for \( p > p_0 \) and \( (\sigma_n)_n \) is divergent.

**Problem 18.4.** Prove that if the series \( \sum_{n=2}^{\infty} |x_n - x_{n-1}| \) is convergent then the sequence \( (x_n)_n \) is convergent. Is the reciprocal satisfied?

**Solution:** If \( s_n = \sum_{k=2}^{\infty} |x_k - x_{k-1}| \), then \( |x_m - x_n| \leq s_m - s_n \), so the sequence \( (x_n)_n \) is Cauchy. The counter-example \( x_n = (-1)^n \frac{n}{n} \) shows the reciprocal is false.

**Problem 18.5.** Consider the sequences \( (a_n)_n, (b_n)_n, (c_n)_n \), such that \( a_n \leq b_n \leq c_n \), for all \( n \). If the series \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} c_n \) are convergent then the series \( \sum_{n=1}^{\infty} b_n \) is also convergent.

**Solution:** Using \( 0 \leq b_n - a_n \leq c_n - a_n \) and the fact that \( \sum_{n=1}^{\infty} (c_n - a_n) \) is convergent, shows that the series \( \sum_{n=1}^{\infty} (b_n - a_n) \) is convergent. Hence \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (b_n - a_n) + \sum_{n=1}^{\infty} a_n \) is convergent.

**Problem 18.6.** Let \( (p_n) \) be the increasing sequence of prime positive integers. Prove that the series \( \sum_{n=1}^{\infty} \frac{1}{p_n} \) is divergent.

**Solution:** Denote by \( \sum_{k<n}^{\prime} \frac{1}{k} \) the sum after all the positive integers free of squares. Since every integer is the product between a square and a number free of squares, for any \( n \geq 1 \), \( \left( \sum_{k<n}^{\prime} \frac{1}{k} \right) \left( \sum_{j<n}^{\prime} \frac{1}{j^2} \right) \geq \sum_{m<n}^{\prime} \frac{1}{m} \). But \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent and \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent, so \( \sum_{k<n}^{\prime} \frac{1}{k} \) is divergent. Suppose \( \sum_{i=1}^{\infty} \frac{1}{p_i} \) is convergent. Then
\[
\exp \left( \sum_{i=1}^{\infty} \frac{1}{p_i} \right) > \exp \left( \sum_{i=1}^{n} \frac{1}{p_i} \right) \prod_{i=1}^{n} \exp \left( \frac{1}{p_i} \right) > \prod_{i=1}^{n} \left( 1 + \frac{1}{p_i} \right) > \sum_{k}^{\infty} \frac{1}{k} = \infty
\]
1. CONVERGENCE

Problem 18.7. Let \( p, q \) be fixed real numbers. Study the convergence of the series \( \sum_{n=1}^{\infty} a_n \), where \( a_n = \frac{1}{n^p \ln^q n} \).

Solution: Let \( b_n = \frac{1}{n^{(p+1)/2}} \). We distinguish the cases

- \( p > 1 \). Since \( \lim_{n \to \infty} \frac{b_n}{a_n} = \infty \) and \( \sum_{n=1}^{\infty} b_n \) is convergent, \( \sum_{n=1}^{\infty} a_n \) is also convergent.

- \( p < 1 \). Since \( \lim_{n \to \infty} \frac{b_n}{a_n} = 0 \) and \( \sum_{n=1}^{\infty} b_n \) is divergent, \( \sum_{n=1}^{\infty} a_n \) is also divergent.

- \( p = 1 \). We use the integral test with \( f(x) = \frac{1}{x \ln^q x} \), which is a decreasing function for \( x > e^{1-q} \), and see that the series is convergent if and only if \( q > 1 \).

Problem 18.8. Let \( (a_n) \) be a decreasing sequence of positive numbers such that the series \( \sum_{n=1}^{\infty} a_n \) is convergent. Then \( \lim_{n \to \infty} na_n = 0 \).

Solution: Let \( s_n = \sum_{k=1}^{n} a_k \). Since \( (s_n)_n \) is convergent, \( (s[k/2])_n \) is also convergent with the same limit. Then \( 0 \leq na_n \leq (n-[n/2])a_n \leq \sum_{k=[n/2]+1}^{n} a_k = s_n - s[k/2] \to 0 \) finishes the proof.

Problem 18.9. Let \( (a_n) \) be a decreasing sequence of positive terms such that \( \lim_{n \to \infty} \frac{2^n a_2^n}{a_n} < 1 \). Then the series \( \sum_{n=1}^{\infty} a_n \) is convergent.

Solution: There is an integer \( N \) and \( 0 < c < 1 \) such that for \( n \geq N \), \( 2^n a_2^n < ca_n \). Then \( a_{2^n+1} + a_{2^n+2} + ... + a_{2^{n+1}} \leq 2^n a_n < ca_n \). Denoting \( S_n = \sum_{k=1}^{n} a_n \) we have \( S_{2^n} - S_{2^n} < ca_n \). Then \( S_n \leq S_{2^n+1} = (S_{2^n+1} - S_{2^n}) + (S_{2^n} - S_{2^n-1}) + ... + (S_{2^{n+1}} - S_{2^n}) + S_{2^n} \leq c(a_n + a_{n-1} + ... + a_N) + S_{2^n} \leq cS_n + S_N \) which gives \( S_n \leq \frac{1}{1-c} S_N \) for any \( n \geq N \). The sequence \( (S_n) \) is increasing and bounded, therefore convergent.

Problem 18.10. Study the convergence of the series \( \sum_{n=1}^{\infty} \frac{e^n n!}{n^n} \).

Solution: Denoting \( a_n = \frac{e^n n!}{n^n} \) we have

\[
n \left( \frac{a_n}{a_{n+1}} \right) = n \left( 1 + \frac{1}{n} \right)^n - 1 \frac{n}{\ln \left( 1 + \frac{1}{n} \right)} - 1 \frac{1}{n}
\]
Since \( \lim_{n \to \infty} \ln \left( 1 + \frac{1}{n} \right)^n = 1 \) the limit of the first fraction is 1. On the other side

\[
\lim_{n \to \infty} \frac{\ln \left( 1 + \frac{1}{n} \right)^n - 1}{n} = \lim_{x \to 0} \frac{\ln(1 + x) - x}{x^2} = -\frac{1}{2}, \text{ so } \lim_{n \to \infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right) = -\frac{1}{2} < 1.
\]

The Raabe-Duhamel test for series shows the series is divergent to \( \infty \).

**Problem 18.11.** Study the convergence of the series \( \sum_{n=1}^{\infty} \left( e - \left( 1 + \frac{1}{n} \right)^n \right)^p \), \( p > 1 \) and \( \sum_{n=1}^{\infty} \left( e - \left( 1 + \frac{1}{n} \right)^n \right)^n \).

**Solution:** From \( \left( 1 + \frac{1}{n} \right)^n < e < \left( 1 + \frac{1}{n} \right)^{n+1} \) we get \( 0 < e - \left( 1 + \frac{1}{n} \right)^n < \frac{1}{n} \left( 1 + \frac{1}{n} \right)^n < e^{-n} \). Therefore both series have positive terms bounded respectively by \( \frac{e^p}{n^p} \) and \( \frac{e^n}{n^n} \), hence they converge.

**Problem 18.12.** Let \( \sum_{n=1}^{\infty} x_n \) be a convergent series with strictly positive terms.

Prove that

(i) \( \lim_{n \to \infty} x_1 + 2x_2 + \ldots + nx_n = 0. \)

(ii) \( \lim_{n \to \infty} nx_n = 0 \)

(iii) \( \sum_{n=1}^{\infty} \frac{x_1 + 2x_2 + \ldots + nx_n}{n(n+1)} = \sum_{n=1}^{\infty} x_n \)

**Solution:** (i) The partial sums \( s_n = \sum_{k=1}^{n} x_k \) converge to the sum \( s \) of the series. Let

\[
t_n = \sum_{k=1}^{n} s_k.
\]

Using the theorem of Cesaro-Stolz and \( \lim(t_{n+1} - t_n) = \lim s_n = s \) we have \( \lim_{n \to \infty} \frac{t_n}{n} = s. \) But \( x_1 + 2x_2 + \ldots + nx_n = s_n + (s_n - s_1) + \ldots + (s_n - s_{n-1}) = s_n - \frac{t_n}{n} \to s - s = 0 \)

(ii) Let \( a_n = x_1 + 2x_2 + \ldots + nx_n \) and \( b_n = n. \) We use (i) and Cesaro-Stolz lemma.

(iii) We have \( \sum_{n=1}^{m} \frac{x_1 + 2x_2 + \ldots + nx_n}{n(n+1)} = \sum_{n=1}^{m} \frac{1}{n(n+1)} \sum_{k=1}^{n} kx_k = \sum_{k=1}^{m} kx_k \sum_{n=k}^{m} \frac{1}{n(n+1)} = \sum_{k=1}^{m} x_k - \sum_{k=1}^{m} \frac{x_1 + 2x_2 + \ldots + mx_m}{m}. \) We use (i) to finish the proof.

**Problem 18.13.** Prove that the series \( \sum_{n=2}^{\infty} \frac{\sin n}{n} \) is convergent. Does it converge absolutely?
Solution: For the first part use Dirichlet’s Test for convergence. Since $\sin x$ and $\sin(x + 1)$ are never zero simultaneously, the function $|\sin x| + |\sin(x + 1)|$ is always positive. Because it is continuous and periodic, there is a positive $m$ such that $|\sin x| + |\sin(x + 1)| \geq m$ for all real $x$. Hence

$$\sum_{n=2}^{\infty} \frac{|\sin n|}{\ln n} = \sum_{k=1}^{\infty} \frac{1}{\ln(2k+1)}.$$ 

Thus the series does not converge absolutely.

Problem 18.14. Given a sequence $(a_n)$ we define $a_n^+ = \max(a_n, 0) = \frac{a_n + |a_n|}{2}$ and $a_n^- = \min(a_n, 0) = \frac{a_n - |a_n|}{2}$.

(a) If $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, show that both of the series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent.

(b) If $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, show that both of the series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are divergent.

Solution: (a) The series $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent as linear combinations of the convergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$.

(b) Suppose $\sum_{n=1}^{\infty} a_n^-$ is convergent. Then $\sum_{n=1}^{\infty} a_n = 2 \sum_{n=1}^{\infty} a_n^- + \sum_{n=1}^{\infty} |a_n|$ is convergent. Contradiction. Similar for $\sum_{n=1}^{\infty} a_n^+$. 

Problem 18.15. Prove that if $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series and $r$ is a real number, then there is a rearrangement of $\sum_{n=1}^{\infty} a_n$ whose sum is $r$.

Solution: Take just enough terms of $\sum_{n=1}^{\infty} a_n^+$ so that their sum is greater than $r$. Then add just enough terms of $\sum_{n=1}^{\infty} a_n^-$ so that the cumulative sum is less than $r$. We continue in this manner using the fact that $\lim a_n = 0$. 

Problem 18.16. What is wrong in the following calculation?

\[ 0 = 0 + 0 + 0 \ldots \\
= (1 - 1) + (1 - 1) + (1 - 1) \ldots \\
= 1 - 1 + 1 - 1 + 1 - 1 \ldots \\
= 1 + (-1 + 1) + (-1 + 1) + \ldots \\
= 1 + 0 + 0 + \ldots = 1 \]

(Guido Ubaldus thought that this proved the existence of God because “something was created out of nothing”).

Solution: The series \( \sum_{n=0}^{\infty} (-1)^n \) is not absolutely convergent, so we can not rearrange its terms. More precisely, the second line is not equal to the third one.

Problem 18.17. Consider the series \( S \) whose terms are the reciprocals of the positive integers that can be written in base 10 notation without using the digit 0. Show that this series is convergent and the sum is less than 90.

Solution: We have \( S = S_1 + S_2 + S_3 + \ldots \), where \( S_n \) is the sum of reciprocals of the \( 9^n \) integers that can be written in base 10 notation using \( n \) digits from the set \( \{1,2,3,\ldots,9\} \). But each such number is greater than \( 10^{n-1} \), so \( S_n < \frac{9^n}{10^{n-1}} \). Therefore \( S < \sum_{n=1}^{\infty} \frac{9^n}{10^{n-1}} = 90 \).

Problem 18.18. Let \( \sum a_n \) be a series with positive terms. Suppose that the sequence \( (r_n) \) defined by \( r_n = \frac{a_{n+1}}{a_n} \) is convergent and \( l = \lim_{n \to \infty} r_n < 1 \), so \( \sum a_n \) converges. Let \( R_n = \sum_{k=n+1}^{\infty} a_k \).

(a) If \( (r_n) \) is a decreasing sequence and \( r_{n+1} < 1 \), show that \( R_n \leq \frac{a_{n+1}}{1-r_{n+1}} \).

(b) If \( (r_n) \) is an increasing sequence, show that \( R_n \leq \frac{a_{n+1}}{1-l} \).

Solution: (a) \( R_n = a_{n+1} + r_{n+1}a_{n+1} + r_{n+1}r_{n+2}a_{n+1} + r_{n+1}r_{n+2}r_{n+3}a_{n+1} + \ldots \leq a_{n+1} + r_{n+1}a_{n+1} + r_{n+1}^2 a_{n+1} + r_{n+1}^3 a_{n+1} + \ldots = \frac{a_{n+1}}{1-r_{n+1}} \).

(b) If \( (r_n) \) is increasing then \( r_n \leq l \), for any \( n \). Then \( R_n = a_{n+1} + r_{n+1}a_{n+1} + r_{n+1}r_{n+2}a_{n+1} + r_{n+1}r_{n+2}r_{n+3}a_{n+1} + \ldots \leq a_{n+1} + l a_{n+1} + l^2 a_{n+1} + l^3 a_{n+1} + \ldots = \frac{a_{n+1}}{1-l} \).

Problem 18.19. Find the values of \( p \) for which the series \( \sum_{n=1}^{\infty} \frac{\ln n}{n^p} \) is convergent.

Solution: Denote \( a_n = \frac{\ln n}{n^p} \) and \( b_n = \frac{1}{n} \). For \( p \leq 1 \), \( \lim_{n \to \infty} \frac{a_n}{b_n} = \infty \), so the series is divergent. If \( p > 1 \) and \( c_n = \frac{1}{n^{1+(p-1)/2}} \), we have \( \lim_{n \to \infty} \frac{a_n}{c_n} = \lim_{n \to \infty} \frac{\ln n}{n^{(p-1)/2}} = 0 \), so the series is convergent.
Problem 18.20. Find the values of p for which the series \( \sum_{n=1}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p} \) is convergent.

Solution: Using the integral test we remark that for \( p = 1 \), the series is divergent. But for \( p \leq 1 \), \( \frac{1}{n \ln n (\ln \ln n)^p} \leq \frac{1}{n \ln n \ln n} \), so the series is also divergent. For \( p > 1 \), the function \( f(x) = \frac{1}{x \ln x (\ln \ln x)^p} \) is decreasing to 0, and the substitution \( u = \ln \ln x \), gives
\[
\int_{3}^{\infty} \frac{1}{x \ln x (\ln \ln x)^p} \, du = \int_{\ln \ln 3}^{\infty} \frac{du}{u^p} = \frac{(\ln \ln 3)^{1-p}}{1-p}.
\]
Therefore the series is convergent.

Problem 18.21. If \( \sum a_n \) is a convergent series with positive terms, is it true that
(a) \( \sum \ln(1 + a_n) \) is convergent?
(b) \( \sum \sin(a_n) \) is convergent?
(c) \( \sum a_n^2 \) is also convergent?
(d) \( \sum a_n b_n \) is also convergent, where \( \sum b_n \) is a convergent series with positive terms?

Solution: (a) \( \lim \frac{\ln(1 + a_n)}{a_n} = 1 \) and \( \ln(1 + a_n) \) is positive for any \( n \), so \( \sum \ln(1 + a_n) \) is convergent.
(b) Since \( \lim a_n = 0 \), there is \( N > 0 \) such that \( a_n \in [0, \pi/2) \) for any \( n > N \). Without any restriction of the generality one can suppose that \( N = 1 \). Then \( \sin(a_n) \geq 0 \), for any \( n \). But \( \lim \frac{\sin(a_n)}{a_n} = 1 \), so \( \sum \sin(a_n) \) is convergent.
(c) Since \( \lim a_n = 0 \), there is \( N > 0 \) such that \( a_n \in [0, 1) \) for any \( n > N \). Without any restriction of the generality one can suppose that \( N = 1 \). Then \( a_n^2 < a_n \), so \( \sum a_n^2 \) is also convergent.
(d) The sequence \( b_n \) is bounded, so there is \( B > 0 \) such that \( b_n < B \), for any \( n \). Then \( a_n b_n < B a_n \), so \( \sum a_n b_n \) is convergent.

Problem 18.22. Show that if \( a_n > 0 \) and \( \lim na_n \neq 0 \), then \( \sum a_n \) is divergent.

Solution: Let \( b_n = \frac{1}{n} \). Since \( \lim \frac{a_n}{b_n} \neq 0 \), and \( \sum b_n \) is divergent, the series \( \sum a_n \) is divergent also.

Problem 18.23. Determine, with proof, the set of real numbers \( x \) for which \( \sum_{n=1}^{\infty} \left( \frac{1}{n} \csc \frac{1}{n} - 1 \right)^x \) converges. [P1988]
It suffices to show that
\[
\frac{1}{n} \csc \frac{1}{n} - 1 = \frac{1}{n} - \sin \frac{1}{n} = \frac{1}{n} - \left( \frac{1}{n} - \frac{1}{6n^3} + \frac{1}{n^5} o(\frac{1}{n}) \right) = \frac{1 + \frac{6}{n^2} o(\frac{1}{n})}{6n^2} \left( 1 - \frac{1}{1 - \frac{1}{n^2} + \frac{1}{n^4} o(\frac{1}{n})} \right)
\]
which shows that the series \( \sum_{n=1}^{\infty} \left( \frac{1}{n} \csc \frac{1}{n} - 1 \right)^2 \) has the same nature like \( \sum_{n=1}^{\infty} \frac{1}{n^2} \). Therefore the series is convergent only for \( x > 1/2 \).

**Problem 18.24.** Test the series \( \sum_{n=2}^{\infty} \frac{1}{(\ln n)^{1/n}} \) for convergence.

**Solution:** For \( n \geq 9 > e^3 \), \( \frac{1}{(\ln n)^{1/n}} < \frac{1}{3n^{1/3}} \), and the comparison test shows the series \( \sum_{n=2}^{\infty} \frac{1}{(\ln n)^{1/n}} \) is convergent.

**Problem 18.25.** Test for convergence \( \sum_{n=1}^{\infty} u_n \), where \( u_n = \int_1^{\infty} e^{-x^n} \, dx \).

**Solution:** We prove the series is divergent. For \( x \in [1, \infty) \) the inequality \( e^{-x^n} \leq e^{-x} \) with \( \int_1^{\infty} e^{-x} \, dx = e^{-1} \) show that the \( \int_1^{\infty} e^{-x^n} \, dx \) is convergent and \( u_n \) is well defined for any \( n \geq 1 \). The substitution \( x = y^{1/n} \) gives \( u_n = \int_1^{\infty} e^{-y^{1/n}} \, dy = \int_1^{\infty} e^{-y^{1/n}} \, dy \geq \frac{1}{n} \int_1^{\infty} e^{-y} \, dy \geq \frac{1}{n} \int_1^{\infty} e^{-y} \, dy = \frac{n}{e} \). The series \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) is divergent, so \( \sum_{n=1}^{\infty} u_n \) is divergent also.

**Problem 18.26.** Prove that if \( \sum_{n=1}^{\infty} a_n \) is a convergent series of positive real numbers, then so is \( \sum_{n=1}^{\infty} (a_n)^{\frac{1}{n+1}} \). [P1988]

**Solution:** It suffices to show that \( \sum_{n=1}^{\infty} (a_n)^{\frac{1}{n+1}} \) is bounded. Denote \( A = \left\{ n \in \mathbb{N}^*, a_n > \frac{1}{2n+1} \right\} \) and \( B = \left\{ n \in \mathbb{N}^*, a_n \leq \frac{1}{2n+1} \right\} \). We remark that \( (a_n)^{\frac{n}{n+1}} < 2a_n \) for \( n \in A \) and \( (a_n)^{\frac{n}{n+1}} \leq \frac{1}{2n} \) for \( n \in B \). Then
\[
\sum_{n=1}^{\infty} (a_n)^{\frac{1}{n+1}} = \sum_{n \in A} (a_n)^{\frac{1}{n+1}} + \sum_{n \in B} (a_n)^{\frac{1}{n+1}} < \sum_{n \in A} 2a_n + \sum_{n \in B} \frac{1}{2n} < 2 \sum_{n=1}^{\infty} a_n + 2
\]

**Problem 18.27.** Let \( (a_n) \) be a sequence of positive reals such that, for all \( n \), \( a_n \leq a_{2n} + a_{2n+1} \). Prove that \( \sum_{n=1}^{\infty} a_n \) diverges. [P1994]

**Solution:** Let \( s_n = a_1 + a_2 + \ldots + a_n \) and \( r_n = a_{2n} + a_{2n+1} + \ldots + a_{2n+1-1} \). Adding the inequalities \( a_{2n+k} + a_{2n+2k+1} \geq a_{2n-1+k}, \ k = 0, 1, \ldots, 2^n - 1 \) we show that...
the sequence \((r_n)_n\) is increasing. Consequently \(r_n \geq r_0 = a_1\). We remark now that
\[s_{2^n - 1} = s_{2^n - 1} + r_n \geq s_{2^n - 1} + a_1\]
which implies \(s_{2^n - 1} \geq na_1\), so \(s_n\) is divergent.

**Problem 18.28.** Let \((a_n)\) be a sequence of positive numbers such that \(\sum a_n\) converges. Find a necessary and sufficient condition for the existence of a sequence of positive numbers \((b_n)\) such that \(\sum b_n\) and \(\sum b_n \sum a_n\) both converge.

**Solution:** The condition is: \(\sum a_n \sqrt{b_n} \) is convergent. The Cauchy-Schwartz inequality
\[
\left(\sum a_n \right) \left(\sum b_n \right) \geq \left(\sum \sqrt{a_n} \right)^2
\]
proves the condition is necessary. The condition is also sufficient, since for \(b_n = \sqrt{a_n}\), the series \(\sum a_n \sqrt{b_n} = \sum b_n = \sum b_n \sqrt{a_n}\) converge.

**Problem 18.29.** Suppose that \(f(x)\) is defined on \([-1, 1]\), and that \(f'''(x)\) is continuous. Show that the series \(\sum_{n=1}^{\infty} (nf(\frac{1}{n}) - nf(-\frac{1}{n}) - 2f'(0))\) converges.

**Solution:** Let \(a_n = nf(\frac{1}{n}) - nf(-\frac{1}{n}) - 2f'(0)\). Since \(f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + o(x^3)\) in a neighborhood of 0, we have \(a_n = o(\frac{1}{n})\). Therefore the series is absolutely convergent.

**Problem 18.30.** Prove the series \(\sum_{n=2}^{\infty} (\ln n)^{-\ln \ln n}\) diverges.

**Solution:** Let \(a_n = (\ln n)^{-\ln \ln n}\) and \(b_n = \frac{1}{n}\). We remark that
\[
\lim_{x \to \infty} \frac{x \ln x}{e^x} = \lim_{x \to \infty} \frac{\ln x}{e^x} = 0.
\]
In particular for \(x = \ln n\) we have \(\lim_{n \to \infty} \frac{b_n}{a_n} = \lim_{n \to \infty} \frac{\ln \ln n}{n} = 0\). Then there is \(n_0\) such that \(a_n \geq b_n\) for all \(n \geq n_0\) and therefore the series \(\sum a_n\) diverges.

**Problem 18.31.** Find all the positive values of \(a\) for which the series \(\sum_{n=1}^{\infty} a^{\ln n}\) converges.

**Solution:** Since \(a^{\ln n} = n^{\ln a}\) the values of \(a\) for which the series converges are \(\ln a < -1\), or \(0 < a < 1/e\).

**Problem 18.32.** Show that for any \(a > 0\) and any \(b\), the series \(\sum_{n=1}^{\infty} \frac{(\ln n)^b}{n^{1+a}}\) converges.
Solution: Let $a_n = \frac{(\ln n)^b}{n^{1+a}}$ and $b_n = \frac{1}{n^{1+a/2}}$. Since $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and the series $\sum_{n=1}^{\infty} b_n$ is convergent, the series $\sum_{n=1}^{\infty} a_n$ is also convergent.

Problem 18.33. Let $a_1, a_2, ..., a_n$ be positive numbers such that $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that there are positive numbers $c_1, c_2, ...$ such that $\lim_{n \to \infty} c_n = \infty$ and $\sum_{n=1}^{\infty} c_n a_n$ is convergent.

2. Sums

Problem 18.34. Evaluate $S = \sum_{p=2}^{\infty} \left( \frac{1}{p} \sum_{q=2}^{\infty} \frac{1}{q^p} \right)$.

Solution: Since all the terms are positive, we can change the order of summation and we obtain $S = \sum_{q=2}^{\infty} f \left( \frac{1}{q} \right)$, where $f(x) = \sum_{p=2}^{\infty} \frac{x^p}{p} = -x - \ln(1 - x)$. The sequence $S_n = \sum_{q=2}^{n} f \left( \frac{1}{q} \right) = 1 + \ln n - \sum_{q=1}^{n} \frac{1}{q}$ is convergent to $1 - C$, where $c$ is Euler’s number, so $S = 1 - C$.

Problem 18.35. Evaluate $S = \sum_{k=1}^{\infty} \frac{1}{2k^2 + 3k}$.

Solution: We transform successively the partial sums

$$S_n = \sum_{k=1}^{n} \frac{1}{2k^2 + 3k} = \frac{2}{3} \sum_{k=1}^{n} \left( \frac{1}{2k} - \frac{1}{2k + 3} \right)
= \frac{2}{3} \left[ \frac{4}{3} + \left( \sum_{k=1}^{n} \frac{1}{k} - \ln n \right) + \frac{1}{2n + 2} - \left( \sum_{k=1}^{2n+3} - \ln(2n + 3) \right) + \ln \frac{n}{2n + 3} \right].$$

Then $S = \lim_{n \to \infty} S_n = \frac{8}{9} - \frac{2}{3} \ln 2$.

Problem 18.36. A sequence $(a_n)$ is defined recursively by the equations $a_0 = a_1 = 1$ and $n(n-1)a_n = (n-1)(n-2)a_{n-1} - (n-3)a_{n-2}$. Find the sum of the series $\sum_{n=0}^{\infty} a_n$.

Solution: One prove by induction that $a_n = \frac{2}{n!}$ for any $n \geq 2$. Then $\sum_{n=0}^{\infty} a_n = 2e - 2$.
PROBLEM 18.37. If \( p > 1 \), evaluate the expression
\[
1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots
\]
\[
1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \ldots
\]

Solution: If we note the expression by \( C \), then
\[
C = 1 + \frac{2}{2^p} C,
\]
so
\[
C = \frac{1}{1 - 2^{1-p}}.
\]

PROBLEM 18.38. Find the sum of the series
\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \ldots
\]
where the terms are the reciprocals of the positive integers whose only prime factors are 2s and 3s.

Solution: The series is the product
\[
\left( \sum_{n=0}^{\infty} \frac{1}{2^n} \right) \left( \sum_{n=0}^{\infty} \frac{1}{3^n} \right) = 2 \cdot \frac{3}{2} = 3
\]

PROBLEM 18.39. Find the interval of convergence of \( \sum_{n=0}^{\infty} n^3 x^n \) and find its sum.

Solution: Using the ratio test gives the radius of convergence \( R = 1 \). For \( x = -1 \) and \( x = 1 \) the series is divergent, so the interval of convergence is \((-1, 1)\).

To find the sum we are looking first after the constants \( a, b, c \) such that
\[
n^3 = (n+1)(n+2)(n+3) + a(n+1)(n+2) + b(n+1) + c
\]
for any \( n \geq 0 \). Identifying the coefficients, we find \( a = -6 \), \( b = 7 \) and \( c = -1 \).

Therefore
\[
\sum_{n=0}^{\infty} n^3 x^n = \sum_{n=0}^{\infty} (n+1)(n+2)(n+3)x^n - 6 \sum_{n=0}^{\infty} (n+1)(n+2)x^n + 7 \sum_{n=0}^{\infty} (n+1)x^n - \sum_{n=0}^{\infty} x^n.
\]
Denote \( f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \). Then
\[
\sum_{n=0}^{\infty} n^3 x^n = f^{(3)}(x) - 6f'(x) + 7f(x) - x^3 + 4x^2 + x \left( \frac{1}{x-1} \right)^3.
\]

PROBLEM 18.40. If \( f(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+n} m^n x^n}{m! n!} \) and \( g(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{m^n x^n}{m! n!} \), prove that \( (f(e))^{(1)}(g(e))^{(1)} = 1 \).

Solution: Both double series are absolutely convergent for any \( x \) so we can change the order of summation without changing the value. So,
\[
\frac{(mx)^n}{n!} = \sum_{m=0}^{\infty} \frac{e^{mx}}{m!} = e^{cx}. \quad \text{Similarly,} \quad \frac{(-mx)^n}{n!} = \sum_{m=0}^{\infty} \frac{(-m)^n x^n}{n!} = e^{-e^{-x}}.
\]
The property to prove follows immediately.
Problem 18.41. Find the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} \).

**Solution:** For \( x = 0 \) the sum of the series is obviously 0. To find the sum for values of \( x \) different from \( n\pi \), we use the identity \( \tan y = \cot y - 2 \cot 2y \). We have

\[
\sum_{n=1}^{m} \frac{1}{2^n} \tan \frac{x}{2^n} = \sum_{n=1}^{m} \left( \frac{1}{2^n} \cot \frac{x}{2^n} - \frac{1}{2^{n-1}} \cot \frac{x}{2^{n-1}} \right) = \frac{1}{2^n} \cot \frac{x}{2^n} - \cot x.
\]

Since

\[
\lim_{n \to \infty} \frac{1}{2^n} \cot \frac{x}{2^n} = \frac{1}{x},
\]

the sum of the series is \( \frac{1}{x} - \cot x \).

Problem 18.42. Let \( B(n) \) be the number of ones that appear in the base two expression for the positive integer \( n \). Evaluate \( \sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)} \).

**Solution:** Consider the sums of the even, respectively odd terms,

\[
\begin{align*}
A &= \sum_{n \text{ even}} B(n) \frac{1}{n(n+1)} = \sum_{m=1}^{\infty} \frac{B(2m)}{2m(2m+1)}, \quad \text{respectively } B = \sum_{n \text{ odd}} B(n) \frac{1}{n(n+1)} = \sum_{m=0}^{\infty} \frac{B(2m+1)}{(2m+1)(2m+2)}.
\end{align*}
\]

Then \( A + B = \sum_{m=1}^{\infty} \frac{1}{(2m+1)(2m+2)} + \sum_{m=1}^{\infty} \frac{1}{2m(2m+1)} \).

The series has then the sum

\[
\sum_{m=1}^{\infty} \frac{1}{(m+1)(2m+1)} = 2 \ln 2.
\]

Problem 18.43. For \( x > 0 \) evaluate \( \sum_{n=1}^{\infty} \frac{(-1)^{[2^n x]}}{2^n} \).

**Solution:** The sum is \( 1 - 2\{x\} \).

Problem 18.44. Evaluate \( S = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2n + mn^2 + 2mn} \).

**Solution:** For any \( n \), we have

\[
\sum_{m=1}^{\infty} \frac{1}{m^2n + mn^2 + 2mn} = \frac{1}{n(n+2)} \sum_{k=1}^{n+2} \frac{1}{k}.
\]

So

\[
S = \sum_{n=1}^{\infty} \left[ \frac{1/2}{n} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} \right) - \frac{1/2}{n+2} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n+2} \right) \right] + \sum_{n=1}^{\infty} \frac{1/2}{n} \left( \frac{1}{n+1} + \frac{1}{n+2} \right) = \frac{7}{4}.
\]

Problem 18.45. For \( x > 1 \) determine the sum of the series

\[
S(x) = \sum_{n=0}^{\infty} \frac{x^{2^n}}{(x+1)(x^2+1)\ldots(x^{2^n}+1)}
\]
Solution: Let $S_1(x) = \sum_{n=0}^{\infty} \frac{1}{(x+1)(x^2+1)\ldots(x^{2n}+1)}$. Both series $S(x)$ and $S_1(x)$ are convergent, and $S(x) + S_1(x) = 1 + S_1(x)$. So $S(x) = 1$.

**Problem 18.46.** Let $x > 1$. Find the sum of the series $S = \sum_{n=0}^{\infty} \frac{2^n}{x^{2n}+1}$.

**Solution:** Using, for example, the comparison with the series $\sum_{n=0}^{\infty} 2^{-n}$, one can see that the series $S = \sum_{n=0}^{\infty} \frac{2^n}{x^{2n}+1}$ and $S_1 = \sum_{n=0}^{\infty} \frac{2^n}{x^{2n}-1}$ are convergent. But

$$S_1 - S = \sum_{n=0}^{\infty} \frac{2^{n+1}}{x^{2n+1}-1} = S_1 - \frac{1}{x-1},$$

so $S = \frac{1}{x-1}$.

**Problem 18.47.** Evaluate $\sum_{n=0}^{\infty} \frac{\cos n}{2^n}$.

**Solution:** We evaluate more generally, $S_1 = \sum_{n=0}^{\infty} \frac{\cos n}{a^n}$ and $S_2 = \sum_{n=0}^{\infty} \frac{\sin n}{a^n}$, for any $a > 1$.

We have $S_1 + iS_2 = \sum_{n=0}^{\infty} \left(\cos 1 + i \sin 1\right)^n = \frac{a}{a - \cos 1 + i \sin 1}$. Therefore $S_1 = \frac{a(a - \cos 1)}{(a^2 + 1) - 2a \cos 1}$ and $S_2 = -\frac{a \sin 1}{(a - \cos 1)^2 + \sin^2 1}$.

**Problem 18.48.** Evaluate $\sum_{n=1}^{\infty} \frac{n^2 + 3n + 1}{(n^2 + n)^2}$.

**Solution:** Since $\frac{n^2 + 3n + 1}{(n^2 + n)^2} = \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n^2} - \frac{1}{(n+1)^2}$, the sum of the series is 2.

**Problem 18.49.** Evaluate the series $\sum_{n=0}^{\infty} \frac{n}{(n+1)!}$.

**Solution:** $\sum_{n=0}^{\infty} \frac{n}{(n+1)!} = \sum_{n=0}^{\infty} \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) = 1$.

**Problem 18.50.** Evaluate $\sum_{n=2}^{\infty} \ln \frac{n^3 - 1}{n^3 + 1}$.

**Solution:** $\prod_{n=1}^{m} \frac{n^3 - 1}{n^3 + 1} = \prod_{n=1}^{m} \frac{n-1}{n+1} \frac{n(n-1)n+1}{n(n+1)+1} = \frac{2 \cdot 3}{m(m+1)(m^2 + m + 1)}$, so the sum of the series is $-\infty$. 
Problem 18.51. If \((f_n)\) is the Fibonacci sequence defined by \(f_0 = 1\), \(f_1 = 1\) and \(f_{n+1} = f_n + f_{n-1}\), evaluate the series \(\sum_{n=1}^{\infty} \frac{1}{f_{n-1}f_{n+1}}\) and \(\sum_{n=1}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}}\).

Solution: We have \(\sum_{n=1}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = \sum_{n=1}^{\infty} \left( \frac{1}{f_{n-1}f_n} - \frac{1}{f_nf_{n+1}} \right) = \frac{1}{f_0} = 1\).

Also \(\sum_{n=1}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = \sum_{n=1}^{\infty} \left( \frac{1}{f_{n-1}f_n} + \frac{1}{f_nf_{n+1}} \right) = \frac{1}{f_0} = 1\).

Problem 18.52. Let \(k \geq 2\) be a fixed integer. For \(n \geq 1\) define \(a_n = 1\), if \(n\) is not a multiple of \(k\), and \(a_n = 1 - k\), if \(n\) is a multiple of \(k\). Evaluate \(\sum_{n=1}^{\infty} a_n/n\).

Solution: If \(S_n = \sum_{i=1}^{n} a_i/i\), then \(S_{nk} = \sum_{i=1}^{nk} a_i/i = \sum_{i=1}^{nk} 1/i - \sum_{i=1}^{nk} k/i = T_{nk} - T_n + \ln k\), where \(T_n = \sum_{i=1}^{n} 1/i - \ln n\). The sequence \((T_n)_n\) is convergent, so \((S_{nk})_n\) is convergent also and \(\lim_{n \to \infty} S_{nk} = \ln k\). For a given \(m \in \mathbb{N}\), there is \(r \in \{0, 1, \ldots, k - 1\}\) such that \(m = nk + r\), where \(n = \lfloor m/k \rfloor\). Moreover, \(n \to \infty\) if \(m \to \infty\). Then \(S_m = S_{nk} + \frac{1}{nk + 1} + \frac{1}{nk + 2} + \ldots + \frac{1}{nk + r}\) shows that \(\lim_{m \to \infty} S_m = \lim_{m \to \infty} S_{nk} = \ln k\).

Problem 18.53. Evaluate \(\sum_{n=0}^{\infty} n^2/n!\).

Solution: \(\sum_{n=0}^{\infty} n^2/n! = \sum_{n=1}^{\infty} n/(n-1)! = \sum_{n=1}^{\infty} n-1/(n-1)! + \sum_{n=1}^{\infty} 1/(n-1)! = 2e\).

Problem 18.54. Evaluate \(\sum_{n=1}^{\infty} 6^n/(3n+1-2n+1)(3n-2n)\).

Solution: The general term of the series can be written as \(a_n = a_{n+1}\), where \(a_n = \frac{2^n}{3n-2n}\). Since \(a_n\) is convergent to 0, the sum of the series is \(a_1 = 2\).

Problem 18.55. Evaluate \(\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}\).

Solution: \(\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{(n-1)n+1} - \frac{1}{n(n+1)+1} = \frac{1}{2}\).

Problem 18.56. Let \(a_n = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2n+2}, n = 1, 2, \ldots\). Does the series \(\sum_{n=0}^{\infty} a_n\) converge, and if so, what is its sum?
2. SUMS

Solution: It is a known result that the sequence defined by $x_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

$\ln n$ is convergent and $\lim x_n$ is Euler’s constant $e \in (0, 1)$. Let $s_n = \sum a_k$. We have

$s_n = \sum_{k=0}^{2n+1} \frac{1}{2k+1} - \frac{1}{2} \sum_{k=1}^{n+1} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{2n+2} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{n+1} \frac{1}{k} = x_{4n+4} + \ln(4n+4) - \frac{1}{2} (x_{2n+2} + \ln(2n+2)) - \frac{1}{2} (x_{n+1} + \ln(n+1)) = x_{4n+4} - \frac{1}{2} x_{2n+2} - \frac{1}{2} x_{n+1} + \frac{3}{2} \ln 2.$

Therefore the series is convergent and its sum is $3 \frac{1}{2} \ln 2$.

Problem 18.57. Evaluate $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

Solution: Let $s_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}$. By the alternating series test the sequence $(s_n)$ is convergent. Consider the sequence $c_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n$ which is convergent to Euler’s number $\gamma$. By the Catalan-Botez identity $s_{2n} = c_{2n} - c_n + \ln 2$, so $s_{2n}$ is convergent to $\ln 2$. Therefore the sum of the series is $\ln 2$.

Problem 18.58. Evaluate the infinite series $S = \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)^3}{n!}$

Solution: It is known that $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1}$. Using the identity $(n+1)^3 = n(n-1)(n-2) + 6n(n-1) + 7n + 1$ we have

$S = 1 - \frac{2^3}{1!} + \frac{3^3}{2!} + \sum_{n=3}^{\infty} (-1)^n \left( \frac{1}{(n-3)!} + \frac{6}{(n-2)!} + \frac{7}{(n-1)!} + \frac{1}{n!} \right) = -\frac{1}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} + 6 \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} - 7 \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = -\frac{1}{4} - e^{-1} + 6(e^{-1} - 1) - 7(e^{-1} - 1 + \frac{1}{1!}) + (e^{-1} - 1 + \frac{1}{1!} - \frac{1}{2!}) = -e^{-1} - 27 \frac{1}{4}$

Problem 18.59. Evaluate the infinite series $\sum_{n=1}^{\infty} \arctan \frac{2}{n^2}$.

Solution: We use the identity $\arctan \frac{x}{1+xy} = \arctan x - \arctan y$ with $x = n + 1$ and $y = n - 1$ and we get

$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} = \lim_{n \to \infty} (\arctan(n+1) + \arctan n - \arctan 1 - \arctan 0) = \frac{3\pi}{4}$

Problem 18.60. Prove that for $|x| < 1, |z| > 1$,

$1 + \sum_{j=1}^{\infty} (1 + x^j)P_j = 0,$
where $P_j$ is

$$
(1 - z)(1 - zx)(1 - zx^2) \cdots (1 - zx^{j-1})

\frac{1}{(z - x)(z - x^2)(z - x^3) \cdots (z - x^j)}.
$$

\[P1990\]

**Solution:** A simple computation proves the identity

\[1 + (1 + u) \frac{1 - z}{z - u} = \frac{1 - zu}{z - u}\]

We will prove by induction

\[1 + \sum_{j=1}^{n} (1 + x^j)P_j = \prod_{j=1}^{n} \frac{1 - zx^j}{z - x^j}\]

For $n = 1$ the identity 2.2 is exactly 2.1 with $u = x$. We suppose 2.2 is true for $n = k$. Then

$$
1 + \sum_{j=1}^{k+1} (1 + x^j)P_j = \prod_{j=1}^{k} \frac{1 - zx^j}{z - x^j} + (1 + x^{k+1})P_{k+1} = \prod_{j=1}^{k+1} \frac{1 - zx^j}{z - x^j} \left(1 + (1 + x^{k+1}) \frac{1 - z}{z - x^{k+1}}\right).
$$

Using again 2.1 this time with $u = x^{k+1}$ we prove 2.2 is satisfied for $n = k + 1$.

It remains to show that

$$
\lim_{n \to \infty} \prod_{j=1}^{n} \frac{|1 - zx^j|}{|z - x^j|} = 0.
$$

Let $t$ be a number in the interval $(\frac{1}{|z|}, 1)$. Since

$$
\lim_{j \to \infty} \frac{|1 - zx^j|}{|z - x^j|} = \frac{1}{|z|} < 1,
$$

there is an $N$ such that $\frac{|1 - zx^j|}{|z - x^j|} < t$ for any $j \geq N$. Then

$$
\lim_{n \to \infty} \prod_{j=1}^{n} \frac{|1 - zx^j|}{|z - x^j|} = \lim_{n \to \infty} \prod_{j=1}^{N-1} \frac{|1 - zx^j|}{|z - x^j|} t^{n-N} = 0
$$

**Problem 18.61.** Let $A$ be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$? \[P2000\]

**Solution:** We remark first that

$$
0 < \sum_{j \geq 0} x_j^2 < \left(\sum_{j \geq 0} x_j\right)^2 = A^2.
$$

We will prove that the set of possible values of $\sum_{j \geq 0} x_j^2$ is the interval $(0, A^2)$. For a given $t \in (0, 1)$ we will construct a sequence $(x_j)_j$ with the properties

\[2.3\]

$$
\sum_{j \geq 0} x_j = A
$$

$$
\sum_{j \geq 0} x_j^2 = A^2 t
$$

Take $(x_j)_j$ a geometric series with first term $x_0$ and ratio $r$. Then the equations 2.3 are equivalent to the system

$$
\begin{align*}
x_0 & = A \\
\frac{x_0}{1 - r} & = A \\
\frac{x_0^2}{1 - r^2} & = A^2 t
\end{align*}
$$
which has the solution \( x_0 = A \frac{2t}{1 + t} \quad r = \frac{1 - t}{1 + t} \).

**Problem 18.62.** Evaluate \( \sum_{n=0}^{\infty} \arccot(n^2 + n + 1) \), where \( \arccot t \) for \( t \geq 0 \) denotes the number \( \theta \) in the interval \( 0 < \theta \leq \pi/2 \) with \( \cot \theta = t \). \[P1986\]

**Solution:** We use the identity \( \arccot \frac{xy + 1}{y - x} = \arccot x - \arccot y \) and we obtain
\[
\sum_{n=0}^{\infty} \arccot(n^2 + n + 1) = \sum_{n=0}^{\infty} (\arccot n - \arccot (n + 1)) = \frac{\pi}{2}
\]

**Problem 18.63.** Evaluate \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{m^2 n}{3^m m^3 + 3^m} \). \[P1999\]

**Solution:** Denote \( x_n = \frac{n}{3^n} \). The double series is then
\[
S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{x_m(x_n + x_m)} = \sum_{m=1}^{\infty} \frac{1}{x_n(x_n + x_m)}
\]
Then \( 2S = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( \frac{1}{x_m x_n + x_m} + \frac{1}{x_n(x_n + x_m)} \right) = \left( \sum_{k=1}^{\infty} \frac{1}{x_k} \right)^2 = \left( \frac{3}{4} \right)^2
\]

**Problem 18.64.** For any positive integer \( n \), let \( \langle n \rangle \) denote the closest integer to \( \sqrt{n} \). Evaluate \( \sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} \).

**Solution:** First we remark that
\[
\langle n \rangle = k \iff k - \frac{1}{2} < n < k + \frac{1}{2} \iff k^2 - k + 1 \leq n \leq k^2 + k
\]
Then
\[
\sum_{n=1}^{\infty} \frac{2^{\langle n \rangle} + 2^{-\langle n \rangle}}{2^n} = \sum_{k=1}^{\infty} 2^{k^2 + k} \frac{2^k + 2^{-k}}{2^n} = \sum_{k=1}^{\infty} (2^k + 2^{-k}) \frac{1}{2^{k^2 + k + 1}} \frac{1}{1 - \frac{1}{2}} =
\]
\[
2 \sum_{k=1}^{\infty} \left( 2^{-(k-1)^2} - 2^{-(k+1)^2} \right) = 3.
\]

**Problem 18.65.** Let the sequence \( a_0, a_1, \ldots \) be defined by the equation
\[
1 - x^2 + x^4 - x^6 + \ldots = \sum_{n=0}^{\infty} a_n (x-3)^n \quad (0 < x < 1).
\]
Find \( \lim \sup_{n \to \infty} |a_n|^\frac{1}{n} \).

**Problem 18.66.** Let \( A = \{(x, y) : 0 \leq x, y < 1\} \). For \( (x, y) \in A \), let
\[
S(x, y) = \sum_{\frac{1}{2} \leq m \leq 2, \frac{1}{2} \leq n \leq 2} x^m y^n,
\]
where the sum ranges over all pairs \((m, n)\) of positive integers satisfying the indicated inequalities. Evaluate
\[
\lim_{(x, y) \to (1, 1), (x, y) \in A} (1 - xy^2)(1 - x^2 y)S(x, y).
\]
\[P1999\]
Solution: We calculate

\[ S(x, y) = \sum_{n \geq 1} y^n \sum_{m \geq n/2} x^m = \sum_{k \geq 0} y^{2k+1} \sum_{m=k+1}^{4k+2} x^m + \sum_{k \geq 1} y^{2k} \sum_{m=k}^{4k} x^m \]

\[ = \sum_{k \geq 0} y^{2k+1} x^{k+1} \frac{x^{3k+2} - 1}{x - 1} + \sum_{k \geq 1} y^{2k} x^k \frac{x^{3k+1} - 1}{x - 1} \]

\[ = \frac{1}{x - 1} \left( \frac{x^3 y}{1 - x^4 y^2} - \frac{xy}{1 - xy^2} + \frac{x^5 y^2}{1 - x^4 y^2} - \frac{xy^2}{1 - xy^2} \right) \]

\[ = 1 + x + y + xy - x^2 y^2 \]

Therefore the desired limit is \( \lim_{(x, y) \to (1, 1)} (1 + x + y + xy - x^2 y^2) = 3 \)

Problem 18.67. Let \((a_n)\) be a decreasing sequence of positive numbers with limit 0 such that \(b_n = a_n - 2a_{n+1} + a_{n+2} \geq 0\) for all \(n\). Prove that \(\sum_{n=1}^{\infty} n b_n = a_1\).

Problem 18.68. Let \((p_n)\) be the increasing sequence of prime positive integers.
Prove that the series \(\sum_{n=1}^{\infty} \frac{1}{p_n}\) is divergent.

Solution: Denote by \(\sum' \frac{1}{k}\) the sum after all the positive integers free of squares. Since every integer is the product between a square and a number free of squares, for any \(n \geq 1\), \(\left( \sum' \frac{1}{k} \right) \left( \sum_{j \leq n} \frac{1}{j^2} \right) \geq \sum_{n \leq m} \frac{1}{m}\).

Problem 18.69. Prove that \(\sum_{n=0}^{\infty} \frac{1}{n!} = e\)
CHAPTER 19

Continuous functions

Problem 19.1. Let \( g, h : \mathbb{R} \to \mathbb{R} \) increasing bijections and \( a > 1 \). Prove that \( f : \mathbb{R} \to \mathbb{R}, f(x) = a^{g(x)} + h(x) \) is bijection.

Solution : Any bijection from \( \mathbb{R} \) to \( \mathbb{R} \) is continuous.

Problem 19.2. Let \( f \) be a continuous real valued function on \([0,1] \times [0,1]\). Let the function \( g \) on \([0,1]\) be defined by \( g(x) = \max\{f(x,y) \mid y \in [0,1]\} \). Prove that \( g \) is continuous.

Problem 19.3. John is leaving the town \( A \) by car at 8:00 am to go to the town \( B \). The next day he is leaving the town \( B \) at 8:00 am to return to \( A \) on the same route. Prove there is a point on which John will be in both days at the same hour.

Problem 19.4. Let \( f : [a,b] \to [a,b] \) be a function such that \( f(f(x)) = x \) and \( f(x) \neq x \), for all \( x \in [a,b] \). Prove that \( f \) has infinitely many discontinuity points.

Solution : We assume that \( f \) doesn’t have infinitely many discontinuity points. We distinguish the cases:

Case 1. If \( f : [a,b] \to [a,b] \) has no discontinuity point, then it has a fixed point. Contradiction.

Case 2. If \( f \) is discontinuous only at one or both of the points \( x = a \) or \( x = b \), then \( f \) is continuous on \((a,b)\) with values on \((a,b)\). The continuous function \( f(x) - x \) is not 0 over this interval, therefore is either always positive, or always negative. Assume \( f(x) - x > 0 \) for all \( x \in (a,b) \) the other case is similar. Then \( x = f(f(x)) > f(x) > x \), contradiction.

Case 3. If \( f \) is discontinuous at several points \( a_1 < a_2 < \ldots < a_n \), then it is continuous on each of the intervals \((a_i,a_{i+1})\), \( 1 \leq i \leq n - 1 \). The image through \( f \) of each of these intervals will be another interval of the same type. If these intervals are the same, then \( f : (a_i,a_{i+1}) \to (a_i,a_{i+1}) \) is continuous and we obtain a contradiction like in the case 2. Since \( f \) is its own inverse, if \( f((a_i,a_{i+1})) = (a_j,a_{j+1}) \), then \( f((a_j,a_{j+1})) = (a_i,a_{i+1}) \), and the intervals can be paired. We have therefore an even number of intervals, which corresponds to an odd number of discontinuity points. But, the image of any discontinuity point through \( f \) is also a discontinuity point. The function \( f \) is its own inverse, so the discontinuity points can also be paired. Since we have an odd number of discontinuity points, at least of these points will be paired with itself, that is there is a point \( x \) such that \( f(x) = x \). Contradiction.

Problem 19.5. Determine all the continuous functions \( f : \mathbb{R} \to \mathbb{R} \) satisfying \( |f(x)| = |x| \), for any \( x \in \mathbb{R} \).
Solution : The function $f$ has constant sign on each of the intervals $(0, \infty)$, $(-\infty, 0)$. Indeed, suppose, for example, there are $x, y \in (0, \infty)$ such that $f(x) > 0$ and $f(y) < 0$. The function $f$ has the property of intermediate values, so there is $z$ between $x$ and $y$, such that $f(z) = 0$. But then $z = 0$, contradiction. Therefore the only solutions are $f_1(x) = |x|$, $f_2(x) = -|x|$, $f_3(x) = x$, $f_4(x) = -x$.

Problem 19.6. Prove that if the function $h : [0, 1) \to \mathbb{R}$ is uniformly continuous, then there exists a unique continuous function $g : [0, 1] \to \mathbb{R}$ such that $g(x) = h(x)$ for all $x \in [0, 1)$.

Solution : Let $(x_n)$ a sequence convergent to $1$. Then $(h(x_n))$ is a Cauchy sequence, hence convergent. Let $(x_n)$ and $(y_n)$ be sequences which are convergent to $1$, and denote $l_1 = \lim_{n \to \infty} h(x_n)$ and $l_2 = \lim_{n \to \infty} h(y_n)$. Let $\epsilon$ be a positive number. Since $h$ is uniformly continuous, there is $\delta$ such that $|h(x) - h(y)| < \epsilon$ for $|x - y| < \delta$. Also there is $N$ such that $|x_n - y_n| < \delta$ for $n > N$. Then $|h(x_n) - h(y_n)| < \epsilon$ for any $n > N$, and consequently $|l_1 - l_2| < \epsilon$. The number $\epsilon$ was arbitrary, so $l_1 = l_2$ and this shows the existence of the limit $l = \lim_{x \to 1} h(x)$. The function $g$ defined by $g(x) = h(x)$ for any $x \in [0, 1)$ and $g(1) = l$ satisfies the requirements. It is easy to see that such a $g$ is unique.

Problem 19.7. A cross-country racer runs a 10-mile race in 50 minutes. Prove that somewhere along the course the racer ran 2 miles in exactly 10 minutes.

Solution : Let $f(t)$ be the distance in miles run by the racer after $t$ minutes form the start of the course. Of course, $f(0) = 0$ and $f(50) = 10$. Let also $g(t) = f(t+10) - f(t) - 2$, $g : [0, 40] \to \mathbb{R}$. The functions $f$ and $g$ are continuous. Suppose there is no time $t$ such that $g(t) = 0$. Then $g$ has constant sign, say $g$ is positive. Adding the inequalities $g(0) > 0, g(10) > 0, g(20) > 0, g(30) > 0, g(40) > 0$ we obtain $0 = f(50) - f(0) - 10 > 0$, contradiction.

Problem 19.8. Show that the set of points where the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \lfloor x \rfloor$ is discontinuous is $\mathbb{Z}$.

Solution : Straightforward.

Problem 19.9. Study the continuity of the function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x$, for $x \in \mathbb{Q}$ and by $f(x) = x^3$, for $x \in \mathbb{R} \setminus \mathbb{Q}$.

Solution : Let $a$ be a point where $f$ is continuous and $r_n$, respectively $s_n$, a sequence of rational, respectively irrational, numbers convergent to $a$. Passing at the limit in the definition we get $a = \lim_{n \to \infty} f(r_n) = \lim_{n \to \infty} f(s_n) = a^3$. So the continuity points of $f$ are $-1, 0, 1$.

Problem 19.10. Let $f : \mathbb{R} \to \mathbb{R}$, be the function defined by $f(x) = (1+x^2)\text{sgn}x$. Show that $f$ is one-to-one, continuous on $\mathbb{R}^*$ and that the inverse $f^{-1} : f(\mathbb{R}) \to \mathbb{R}$ is continuous.

Solution : Straightforward.

Problem 19.11. Let $f : (a, b) \to \mathbb{R}$ a continuous function such that the limits $f(a+)$ and $f(b-)$ exists and are finite. Show that $f$ is bounded.
**Solution:** The function $g : [a, b] \to \mathbb{R}$, defined by $g(x) = f(x)$ for $x \in [a, b]$ and $g(a) = f(a+)$, $g(b) = f(b-)$, is continuous one the compact $[a, b]$, hence bounded.

**Problem 19.12.** Give an example of a continuous bounded function $f : (a, b) \to \mathbb{R}$, such that the limit $f(a^+)$ doesn’t exist.

**Solution:** $f(x) = \sin \frac{1}{x-a}$.

**Problem 19.13.** Let $f : \mathbb{R} \to \mathbb{R}$ continuous such that there is a fixed point of $f \circ f$. Then $f$ has also a fixed point.

**Solution:** Suppose $f$ has no fixed points. The equation $g(x) = f(x) - x = 0$ has no solution, and since $g$ is continuous, then has constant sign, say $g(x) < 0$ (the case $g(x) > 0$ is similar). Then $f(f(x)) < f(x) < x$, for all $x$. Contradiction.

**Problem 19.14.** Let $A \subset \mathbb{R}$ be such that any continuous function $f : A \to \mathbb{R}$ is bounded. Then $A$ is compact.

**Problem 19.15.** Let $f : [a, b] \to [a, b]$ be continuous. Then $f$ has a fixed point. Application: an elevator is going up at 9 a.m. from the first floor to the 10th of a building. Next day is going down at 9 a.m. from the 10th to the first floor. Prove there is a point where the elevator is at the same hour both days.

**Solution:** The function defined by $g(x) = f(x) - x$ is continuous on $[a, b]$. Since $g(a) \geq 0$ and $g(b) \leq 0$ the function $g$ has at least a zero.

**Problem 19.16.** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic. Then $f$ is bounded and has at least a fixed point.

**Solution:** If $T$ is a period of $f$, then $f([0, T])$ is bounded. Then $\lim_{x \to -\infty} (f(x) - x) = \infty$ and $\lim_{x \to \infty} (f(x) - x) = -\infty$. Then function $f(x) - x$ takes positive and negative values, hence has a zero.

**Problem 19.17.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Prove there is $c \in (a, b)$ such that $f(c) = \frac{1}{a-c} + \frac{1}{b-c}$.

**Solution:** Consider the function $g(x) = f(x) - \frac{1}{a-x} - \frac{1}{b-x}$ and evaluate its limits towards $a$ and $b$.

**Problem 19.18.** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, periodic and non-constant function. Prove that the limits $f(+\infty)$ and $f(-\infty)$ doesn’t exist.

**Solution:** Let $T > 0$ be a period of $f$ and $a, b \in \mathbb{R}$ such that $f(a) \neq f(b)$. The sequences $x_n = a + nT$ and $y_n = b + nT$ are converge to $\infty$ and $f(x_n) = f(a) \neq f(b) = f(y_n)$.

**Problem 19.19.** Let $f : [a, b] \to \mathbb{R}$ be semiconvex and $x_0 \in (a, b)$. If the limits $f(x_0^+)$ and $f(x_0^-)$ exist, then $f$ is continuous at $x_0$. 

Solution: By hypothesis $f\left(\frac{x+y}{2}\right) \leq \frac{f(x) + f(y)}{2}$. With $x = x_0 + \frac{1}{n}$ and $y = x_0 - \frac{1}{n}$ we have $f(x_0) \leq \frac{f(x_0 +) + f(x_0 -)}{2}$. For $x = x_0$ and $y = x_0 + \frac{1}{n}$ we get $f(x_0 +) \leq f(x_0)$ and similarly $f(x_0 -) \leq f(x_0)$.

Problem 19.20. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) = f(b)$, and let $m = \inf f$ and $M = \sup f$. Then $f$ takes two times every value in $[m, M]$.

Problem 19.21. The function $f : [0, \infty[ \to \mathbb{R}$, defined by $f(x) = \sqrt{x}$, is uniformly continuous.

Problem 19.22. The function $f : (0, 1) \to \mathbb{R}$, $f(x) = \sin \frac{\pi}{x}$, is not uniformly continuous.

Problem 19.23. The function $f : (0, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x} \ln(1+x)$, is uniformly continuous.

Problem 19.24. The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x^2$, is not uniformly continuous.

Problem 19.25. Prove that the product of two uniformly continuous bounded functions is uniformly continuous.

Problem 19.26. Let $f : (a, b) \cup (b, c) \to \mathbb{R}$ be a continuous function. If the lateral limits of $f$ at $a, b$ and $c$ exist and are finite, then $f$ is uniformly continuous. Prove that the reciprocal is also true.

Problem 19.27. Let $f : \mathbb{R} \to \mathbb{R}$ be a non-constant function and $T > 0$ such that $f(x + rT) = f(x)$ for any rational number $r$. Prove that $f$ is discontinuous in every $x \in \mathbb{R}$.

Problem 19.28. Let $f : \mathbb{R} \to \mathbb{R}$ a function which is unbounded on any open interval. Prove that $f$ is discontinuous on every real x.
CHAPTER 20

The intermediate value property

Definition 20.1. Let $I$ an interval and $f : I \to \mathbb{R}$ a function. We say $f$ has the intermediate value property (IVP) if for any $a, b \in I$, $a < b$, and $\lambda$ between $f(a)$ and $f(b)$, there is $c \in (a, b)$ such that $f(c) = \lambda$.

Problem 20.1. Let $f : I \to \mathbb{R}$ be a function. Then $f$ has IVP if and only if $f(J)$ is an interval for any interval $J \subset I$.

Problem 20.2. If $f$ is a function with the intermediate value property such that $f \circ f$ is injective, then $f \circ f$ is increasing.

Solution: From $f \circ f$ injective follows $f$ injective and having the IVP it will be monotonic. Then $f \circ f$ is increasing.

Problem 20.3. Give an example of two functions $f, g$ with IVP such that $f + g$ doesn't have IVP.

Problem 20.4. Prove there are no continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x)$ is rational if and only if $f(x + 1)$ is irrational, for every $x \in \mathbb{R}$.

Problem 20.5. If $f : I \to \mathbb{R}$ is a monotonic function, then $f$ has lateral limits in any point $x \in I$.

Problem 20.6. Let $f : [0, 1] \to [0, 1]$ be a function such that $f(0) = 0$, and $f(1) = 1$.

a) Prove that, if $f$ has IVP then $f$ is surjective.

b) Is there any surjective $f$ which doesn’t have IVP?

Problem 20.7. If $f : I \to \mathbb{R}$ is monotonic and has IVP, then $f$ is continuous.

Problem 20.8. Prove that if $f : I \to \mathbb{R}$ is monotonic, and $f(I)$ is an interval then $f$ is continuous.

Problem 20.9. Let $f, g : \mathbb{R} \to \mathbb{R}$ be such that $f$ is continuous, $g$ is monotonic, and $f(x) = g(x)$ for any rational $x$. Prove that $f = g$.

Problem 20.10. a) Let $f, g : \mathbb{R} \to \mathbb{R}$ be periodic functions such that $f + g$ has a limit at $\infty$. Prove that $f + g$ is constant.

b) If $a \cos ax + b \cos bx \geq 0$ for any $x \in \mathbb{R}$, then $a = b = 0$.

Problem 20.11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with IVP, such that $f \circ f$ is injective. Prove that $f \circ f$ is increasing.

Problem 20.12. Let $f : \mathbb{R} \to \mathbb{R}$ a function with the intermediate value property such that $f \circ f$ is injective. Prove that $f \circ f$ is strictly increasing.

Solution: From $f \circ f$ injective, it follows that $f$ is injective. Using Darboux’s theorem, the injective function $f$ with IVP is monotonic. Then $f \circ f$ is increasing.
Problem 20.13. Prove that the equation $\sqrt[3]{x} + a + \sqrt[3]{x} + b = \sqrt[3]{c}$ has a unique real solution.

Solution: The function $f(x) = \sqrt[3]{x} + a + \sqrt[3]{x} + b - \sqrt[3]{c}$ is surjective onto $\mathbb{R}$, since it is continuous and $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to +\infty} f(x) = +\infty$. Then the equation $f(x) = 0$ has a solution. Also the function $f$ is increasing, therefore injective and as a consequence the equation $f(x) = 0$ has a unique solution.
1. Differentiability

Problem 21.1. Let \( f : [-1, 1] \to (0, \infty) \) be a function derivable in 0 such that \( f(0) = 1 \). Determine \( \lim_{n \to \infty} \left[f\left(\frac{1}{n}\right)\right]^n \) and give an example of a function with these properties.

Solution: We have
\[
\left(f\left(\frac{1}{n}\right)\right)^n = \left[1 + \left(f\left(\frac{1}{n}\right) - f(0)\right)\right] \cdot \left(f\left(\frac{1}{n}\right) - f(0)\right) \to e^{f'(0)}
\]
The functions \( \cos x \) or \( 1 - \sin x \), or \( 1 - x^2 \) satisfy the conditions of the problem.

Problem 21.2. Find the antiderivative of the function \( f(x) = 2x + 4 \) with the graph tangent to the line of equation \( L: 6x - y + 3 = 0 \).

Solution: The antiderivatives of \( f \) are of the form \( F(x) = x^2 + 4x + c \), with \( c \) real constant. The graph of \( F \) is tangent to the line \( L : y = 6x + 3 \) if and only if the equation \( x^2 + 4x + c = 6x + 3 \) has a unique solution. This corresponds to the discriminant \( \Delta = 4 - 4(c - 3) = 0 \), that is \( c = 4 \).

2. Applications of the derivatives

21.1. Let \( f : I \to \mathbb{R} \) be an increasing (respectively decreasing) function on an interval \( I \). If \( f \) is derivable on \( I \), then \( f' \geq 0 \) (respectively \( f' \leq 0 \)) on \( I \).

Proof: We can suppose without any cost \( f \) increasing (otherwise consider \(-f\)) on \( I \). Then for any \( x, x_0 \in I \) we have \( \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \) and passing at limit with \( x \to x_0 \) we get \( f'(x_0) \geq 0 \).

21.2. Let \( f : I \to \mathbb{R} \) be a function derivable on the interval \( I \). If \( f' \geq 0 \) (respectively \( f' \leq 0 \)) on \( I \) then \( f \) is increasing (respectively decreasing) on \( I \).

Proof: Suppose \( f' \geq 0 \) (for the other case consider \(-f\)) and let \( x, y \in I \) with \( x \leq y \). From Lagrange's theorem on \([x, y]\), there is \( c \in [x, y] \) such that \( f(y) - f(x) = (y - x)f'(c) \geq 0 \).

21.3. If the function \( f : I \to \mathbb{R} \) is derivable and \( f' = 0 \) on \( I \) then \( f \) is constant.

Proof: From theorem 21.2, the function is increasing and decreasing in the same time, therefore constant.
Corollary 21.4. If \( f \) and \( g \) are two derivable functions on the interval \( I \) and \( f' = g' \), then \( f \) and \( g \) differ by a constant.

**Remark** In all the previous proofs is important to consider the behavior of the function over an interval and not on isolated points. The value of the derivative on one point is not giving any information on the behavior of the function, as one can see in the following example. Consider the function \( f : \mathbb{R} \to \mathbb{R} \), defined by 
\[
f(x) = x + 2x^2 \sin \frac{1}{x}
\]
if \( x \neq 0 \) and \( f(0) = 0 \). The derivative satisfies \( f'(0) = 1 > 0 \), but the function is not increasing or decreasing since on any neighborhood of 0, the function \( f \) has positive and negative values.

**Problem 21.3.** Let \( a \) be a fixed real number. Prove that \( a^x \geq x + 1 \) for any \( x \in \mathbb{R} \) if and only if \( a = e \).

**Solution:** If \( a = e \) then we study the behavior of the function \( f(x) = e^x - x - 1 \). The derivative \( f'(x) = e^x - 1 \) has negative sign on \((-\infty, 0)\) and positive sign on \((0, \infty)\), so the function has a minimum at \( x = 0 \). But \( f(x) \geq f(0) \) can be written \( e^x \geq x + 1 \).

Suppose now \( a^x \geq x + 1 \) for any \( x \). Taking \( x = 0 \) we see \( a \geq 2 \). Consider the function \( f(x) = a^x - x - 1 \). The derivative \( f'(x) = a^x \ln a - 1 \) has the root \( x_0 = -\frac{\ln \ln a}{a} \) which is a minimum for the function \( f \). By hypothesis \( f(x) \geq 0 \) for all \( x \in \mathbb{R} \), therefore \( f(x_0) \geq 0 \). But \( f(x_0) = \frac{(\ln \ln a)(\ln a - 1)}{a \ln a} \leq 0 \). Consequently \( \ln \ln a = \ln a - 1 = 0 \), so \( a = e \).

**Problem 21.4.** Let \( a > 0 \) be a fixed real number. Prove that \( a^x \geq x^a \) for all \( x > 0 \) if and only if \( a = e \).

**Solution:** The inequality is equivalent with \( f(x) \leq f(a) \), for any \( x > 0 \), where \( f(x) = \frac{\ln x}{x} \). Therefore \( x = a \) is a maximum for \( f \), but this function has a maximum only in \( x = e \).

**Problem 21.5.** Prove that for any integer \( n \) and real number \( x \), \( (1 + \frac{x}{n})^n \leq e^x \).

**Solution:** In the inequality \( 1 + y \leq e^y \), take \( y = \frac{x}{n} \) and raise to the power \( n \).

**Problem 21.6.** Let \( a \) be a fixed real number and consider the sequence \( x_n = \left(1 + \frac{a}{n}\right)^n \). Prove that the sequence \( x_n \) is decreasing if and only if \( a \geq \frac{1}{2} \).

**Problem 21.7.** Let \( a \) be a fixed real number and consider the sequence defined by \( x_n = \left(1 + \frac{a}{n}\right)^{n+1} \). Prove that the sequence \( x_n \) is decreasing if and only if \( 0 < a \leq 2 \).

**Problem 21.8.** Consider the sequence defined by \( \left(1 + \frac{1}{n}\right)^{n+x} = e \). Prove that \( x_n \) is decreasing towards the limit \( \frac{1}{2} \).
Solution: By hypothesis $x_n = f(n)$, where $f(x) = \frac{1}{\ln(x+1) - \ln x} - x$. We prove $f$ is increasing on $[1, \infty)$. It suffices to prove $f'(x) = \frac{1}{x(x+1)\ln^2(1+1/x)} - 1 \geq 0$ or equivalently $g(x) = \frac{1}{\sqrt{x(x+1)}} - \ln(x+1) + \ln x \geq 0$. Since $g'(x) = \frac{1}{x(x+1)} \left( 1 - \frac{x+1/2}{\sqrt{x(x+1)}} \right) < 0$, we have $g(x) \geq \lim_{x \to \infty} g(x) = 0$.

For the limit of the sequence it suffices to prove that $\lim_{x \to \infty} f(x) = \frac{1}{2}$, or with $y = \frac{1}{x}$, $\lim_{y \to 0} \frac{1}{\ln(1+y)} - \frac{1}{y} = \frac{1}{2}$, which can be proved easily using l'Hopital's rule or a Taylor expansion.

**Problem 21.9.** Prove that $\left(1 + \frac{1}{n}\right)^{n+\alpha} > e$ for any positive integer $n$ if and only if $\alpha \geq \frac{1}{2}$.

**Solution:** Let $f(x) = \frac{1}{\ln(x+1) - \ln x} - x$. The inequality can be written as $\alpha > f(n)$, and holds for any positive integer $n$ if and only if $\alpha \geq \sup_x f(x)$. But $f$ is increasing, so $\sup_x f(x) = \lim_{x \to \infty} f(x) = \frac{1}{2}$ (see problem 21.8).

### 3. The Derivative

**Problem 21.10.** Solve the equation $2^x + 2\sqrt{1-x^2} = 3$.

**Solution:** Obviously, the solutions of the equation satisfy $x \in [0, 1]$. We prove $x = 0$ and $x = 1$ are the only solutions of the equation. The function $f : [0, 1] \to \mathbb{R}$ is increasing on the interval $[0, \sqrt{2}/2]$ and decreasing on $[\sqrt{2}/2, 1]$. The identity $f(x) = f(\sqrt{1-x^2})$ makes it sufficient to prove $f$ increasing on $[0, \sqrt{2}/2]$. The function $g(x) = \frac{2x}{\sqrt{x^2}}$ is decreasing on $[0, 1]$, since $g'(x) = \frac{2x}{x^2} (x \ln 2 - 1) < 0$. Then $f'(x) = x \ln 2 \left(g(x) - g(\sqrt{1-x^2})\right) > 0$, since $x < \sqrt{1-x^2}$ on $[0, \sqrt{2}/2]$.

**Problem 21.11.** Let $F$ be a real valued continuous function on $[0, \infty)$ such that $\lim_{x \to \infty} \left( f(x) + \int_0^x f(t)dt \right)$ exists. Prove that $\lim_{x \to \infty} f(x) = 0$.

**Solution:** Using l'Hopital’s theorem

$$\lim_{x \to \infty} \left( f(x) + \int_0^x f(t)dt \right) = \lim_{x \to \infty} \frac{\left(e^x \int_0^x f(t)dt \right)'}{e^x} = \lim_{x \to \infty} \int_0^x f(t)dt$$

**Problem 21.12.** Let $f$ be a real function on the real line with continuous third derivative. Prove that there exists a point $a$ such that $f(a) \cdot f'(a) \cdot f''(a) \cdot f'''(a) \geq 0$. [P1998]
We prove first the following result

**Solution:** We prove first the following result

*If* $g$ *is decreasing and concave then* $\lim_{x \to \infty} g(x) = -\infty$. *If* $g$ *is increasing and concave then* $\lim_{x \to -\infty} g(x) = -\infty$.

If $g$ is decreasing then there is $\lim_{x \to \infty} g(x) = l \in [-\infty, \infty)$. Let $y$ be a fixed real number. Since $g$ is concave, for any $x$ the inequality $g\left(\frac{x+y}{2}\right) \geq \frac{g(x) + g(y)}{2}$ holds. Passing at the limit with $x \to \infty$ we obtain $2l \geq l + g(y)$. If $l \neq -\infty$ then $l \geq g(y)$ which is contradictory with the fact $g$ is decreasing. The proof of the second part is similar.

To solve now the problem let us suppose that $f(x) \cdot f'(x) \cdot f''(x) \cdot f'''(x) < 0$ for any $x$. Then $f, f', f'', f'''$ have all constant sign. We can suppose without loss of generality that $f > 0$. We distinguish the following 4 cases:

1. $f' < 0, f'' < 0, f''' < 0$ The function $f$ is decreasing and concave, so $\lim_{x \to \infty} f(x) = -\infty$, contradiction with $f > 0$.

2. $f' > 0, f'' > 0, f''' > 0$ The function $f$ is increasing and concave, therefore $\lim_{x \to -\infty} f(x) = -\infty$, contradiction with $f > 0$.

3. $f' < 0, f'' < 0, f''' < 0$ The function $f$ is decreasing and concave, therefore $\lim_{x \to -\infty} f(x) = -\infty$, contradiction with $f' > 0$.

4. $f' > 0, f'' > 0, f''' > 0$ The function $f$ is increasing and concave, therefore $\lim_{x \to \infty} f(x) = -\infty$, contradiction with $f' > 0$.

**Problem 21.13.** Let $f(x)$ be a positive-valued function over the reals such that $f'(x) > f(x)$ for all $x$. For what $k$ must there exist $N$ such that $f(x) > e^{kx}$ for $x > N$? [P1994]

**Solution:** The condition can be written $g'(x) > 0$, where $g(x) = \ln f(x) - x$ and an equivalent form for $f(x) > e^{kx}$ is

$$
(3.1) \quad g(x) > (k-1)x
$$

For $k \geq 1$, and $g(x) = \frac{k-1}{2} x$ there is no $N$ such that $3.1$ is satisfied for $x > N$. If $k < 1$ then $\lim_{x \to \infty} (g(x) - (k-1)x) = \infty$, so there is an $N$ such that for $x > N$ $3.1$ holds.

**Problem 21.14.** Let $f$ be an infinitely differentiable real-valued function defined on the real numbers. If

$$
\frac{1}{f\left(\frac{1}{n}\right)} = \frac{n^2}{n^2 + 1}, \quad n = 1, 2, 3, \ldots,
$$

compute the values of the derivatives $f^{(k)}(0), k = 1, 2, 3, \ldots$. [P1992]

**Solution:** With $g(x) = f(x) - \frac{1}{1 + x^2}$ we have $g\left(\frac{1}{n}\right) = 0$, $n = 1, 2, 3, \ldots$ By the theorem of Rolle there is a sequence $(x^{(1)}_n)_n$ such that $\frac{1}{n+1} < x^{(1)}_n < \frac{1}{n}$ (*) and $g'(x^{(1)}_n) = 0$. As a consequence of (*) the sequence $x^{(1)}_n$ is decreasing to 0, hence $g'(0) = 0$. Using again Rolle’s theorem there is a sequence $(x^{(2)}_n)_n$ such that $x^{(1)}_{n+1} < x^{(2)}_n < x^{(1)}_n$ (**) and $g''(x^{(2)}_n) = 0$. From (**), $x^{(2)}_n$ is decreasing to 0, and
g^{(2)}(0) = 0. By recurrence we prove in this way that \( g^{(k)}(0) = 0 \), for any \( k \). Thus \( f^{(k)}(0) = \phi^{(k)}(0) \), where \( \phi(x) = \frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \ldots \) and consequently \( f^{(2k+1)}(0) = 0, f^{(2k)}(0) = (-1)^n(2n)! \).

**Problem 21.15.** Suppose \( f \) is a differentiable function from the reals into the reals. Suppose \( f'(x) > f(x) \) for all \( x \in \mathbb{R} \), and \( f(x_0) = 0 \). Prove that \( f(x) > 0 \) for all \( x > x_0 \).

**Solution:** Let \( g(x) = \frac{f(x)}{e^x} \). Then \( g'(x) = \frac{f'(x) - f(x)}{e^x} < 0 \) for all \( x \in \mathbb{R} \), so \( g \) is increasing. Hence \( f(x) = g(x)e^x > g(x_0)e^x = 0 \), for all \( x > x_0 \).

**Problem 21.16.** How many real zeros does \( f(x) = 2^x - 1 - x^2 \) have?

**Solution:** The first and second derivative of \( f \) are \( f'(x) = 2^x \ln 2 - 2x \) and \( f''(x) = 2^x (\ln 2)^2 - 2 \). The second derivative has only the zero \( x_0 = \frac{\ln 2}{\ln 2 - 2} \), which proves \( f'(1) = 2 \ln 2 - 2 < 0 \), \( \lim_{x \to 1} f'(x) = \infty \) the function \( f'(x) \) has only two zeros \( x_1 \in (\infty, 1) \) and \( x_2 \in (1, \infty) \). Moreover \( f'(x) \) is positive on \( (-\infty, x_1) \cup (x_2, \infty) \) and negative on \( (x_1, x_2) \). The function \( f(x) \) is therefore increasing on the intervals \( (-\infty, x_1) \) and \( (x_2, \infty) \), decreasing on \( (x_1, x_2) \). Consequently, \( f \) can have at most 3 zeros, one in each of these intervals. One can guess easily two of these zeros, since \( f(0) = f(1) = 0 \). To show that there is also a third zero it is enough to remark that \( f(4) = -1 \) and \( f(5) = 6 \), so there is a third zero in the interval \((4, 5)\).

**Problem 21.17.** Let \( a_1, a_2, \ldots, a_n \) be real numbers and let \( f(x) = a_1 \sin x + a_2 \sin 2x + \ldots + a_n \sin nx \). Given that \( |f(x)| \leq |\sin x| \) for all real \( x \), prove that \( |a_1 + 2a_2 + \ldots + na_n| \leq 1 \).

**Solution:** \( |a_1 + 2a_2 + \ldots + na_n| = \lim_{x \to 0} \left| \frac{f(x)}{x} \right| \leq 1 \).

**Problem 21.18.** Let \( f \) be a function such that \( f(1) = 1 \) and \( f'(x) = \frac{1}{x^2 + f^2(x)} \) for all \( x \geq 1 \). Show that \( \lim_{x \to \infty} f(x) \) exists and is less than \( 1 + \frac{\pi}{4} \).

**Solution:** The function has positive derivative so is increasing. Then the limit \( \lim_{x \to \infty} f(x) \) exists and \( f'(x) \leq \frac{1}{x^2 + f(1)^2} = \frac{1}{x^2 + 1} \), for any \( x \geq 1 \), which proves that the function \( g(x) = f(x) - \arctan x \) is decreasing. Hence \( g(x) \leq g(1) = 1 - \frac{\pi}{4} \) and consequently \( f(x) \leq \arctan x + 1 - \frac{\pi}{4} \). Therefore \( \lim_{x \to \infty} f(x) \leq \lim_{x \to \infty} \left( \arctan x + 1 - \frac{\pi}{4} \right) = 1 + \frac{\pi}{4} \).

**Problem 21.19.** Let \( f : (-2, 2) \to \mathbb{R} \) be a function of class \( C^2 \) such that \( f(0) = 0 \). Show that the sequence \( (u_n) \) defined by \( u_n = \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) \) is convergent and compute its limit.
Solution: Let \( c_k \in \left(0, \frac{k}{n^2}\right) \) be such that \( f\left(\frac{k}{n^2}\right) - f(0) = \frac{k}{n^2} f'(c_k) \), and \( d_k \in (0,c_k) \) such that \( f'(c_k) - f'(0) = c_k f''(d_k) \). Then \( |u_n - \frac{k}{n^2} f'(0)| = |\sum_{k=1}^{n} \frac{k^2}{n^4} \max_{x \in [0,1]} |f''(x)| \). Passing at limit with \( n \to \infty \) we obtain \( \lim_{n \to \infty} u_n = \frac{f'(0)}{2} \).

4. Taylor expansion

**Problem 21.20.** Let \( a \) be a real number and \( n \geq 3 \) an integer. Evaluate
\[
\lim_{x \to \infty} x(2x - \sqrt[n]{x^n + ax^{n-1}} + 1 - \sqrt[n]{x^n - ax^{n-1}} + 1).
\]

**Solution:** The limit to evaluate is \( \lim_{y \to 0} \frac{2 - \sqrt[n]{y^n + ay + 1} - \sqrt[n]{y^n - ay + 1}}{y^2} \) after
\[ y = \frac{1}{x} \]. Using the Taylor expansion \( \sqrt[n]{1 + u} = 1 + \frac{u}{n} - \frac{n-1}{2n^2} u^2 + o(u^3) \) around \( u = 0 \), we have
\[ 2 - \sqrt[n]{y^n + ay + 1} - \sqrt[n]{y^n - ay + 1} = \frac{n-1}{n^2} 2a^2 y^2 + o(y^3) \]. Hence the limit is \( \frac{2a^2(n-1)}{n^2} \).

**Problem 21.21.** Donner un exemple d’une fonction \( f \) qui admet un développment limité et telle que \( f' \) n’admet pas.

**Solution:** Take \( f(x) = \begin{cases} x^3 \sin \frac{1}{x^2} & , \ x \neq 0 \\ 0 & , \ x = 0 \end{cases} = o(x^2), \) but \( f'(x) \neq o(x) \).

**Problem 21.22.** Seria lui Taylor ptr \( f(x) = \frac{x}{e^x - 1} \).

**Solution** The function \( f \) is extended by continuity with \( f(0) = \lim_{x \to 0} \frac{x}{e^x - 1} = 1 \), and
\[ f'(0) = \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{x - e^x + 1}{x(e^x - 1)} = \lim_{x \to 0} \frac{1 - e^x}{x(e^x - 1) + xe^x} = \lim_{x \to 0} \frac{-e^x}{e^x + e^x + xe^x} = \frac{1}{2} . \]

Deci dezvoltarea Taylor este de forma \( \frac{x}{e^x - 1} = 1 - \frac{1}{2} x + \sum_{k \geq 2} a_k x^k \) (*).

Aratam ca \( a_{2k+1} = 0, \forall k \geq 1 \). Intr-adevar, fie \( \phi(x) = \frac{x}{e^x - 1} - 1 + \frac{1}{2} x \). Avem
\[ \phi(-x) = \frac{-x}{e^{-x} - 1} - 1 - \frac{1}{2} x = \frac{x}{1 - e^x} - 1 - \frac{1}{2} x = \frac{xe^x}{e^x - 1} - 1 - \frac{1}{2} x = \phi(x) \]. Cum \( \phi(x) = \sum_{k \geq 2} a_k x^k \) egalitatea \( \phi(x) = \phi(-x) \) implica \( \sum_{k \geq 2} a_k x^k = \sum_{k \geq 2} a_k(-1)^k x^k \) si de aici \( a_{2k+1} = (-1)^{2k+1} a_{2k+1}, \) deci \( a_{2k+1} = 0 \).
Egalitatea (*) se transpune în 1 = (1 + x/2 + x^2/3! + ...) (1 - 1/2 x + a_2 x^2 + a_3 x^3 + ...).

Facand produsul în partea dreapta și identificând coeficientul lui x^n deducem
\[ a_n + \frac{a_{n-1}}{2!} + \frac{a_{n-2}}{3!} + \ldots + \frac{a_2}{(n-1)!} - \frac{1}{2} \frac{1}{n!} + \frac{1}{(n+1)!} = 0. \]

Luan pe rand \( n = 2, n = 4, \ldots \) săi deducem \( a_2 = \frac{1}{12}, a_4 = -\frac{1}{6}, a_6 = \frac{1}{6} \cdot 7! \) etc.

**Problem 21.23.** Let \( k \) be an integer. Prove that the formal power series
\[ \sqrt{1+kx} = 1 + a_1x + a_2x^2 + \ldots \]
has integral coefficients if and only if \( k \) is divisible by 4.

**Solution:** The coefficients \( a_n \) are given by
\[ a_n = \frac{1}{n!} \left( \frac{1}{2} - \frac{1}{2} - 1 \right)^n. \]
In particular, \( a_2 = -\frac{k^2}{8} \) is integer only if \( k \) is divisible by 4. Suppose now that \( k = 4p \), with \( p \) integer. Then
\[ a_n = (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \ldots (2n-3)}{n!} (2n-3) 2^n p^n = -2(-1)^n \frac{(2n-2)!}{n! (n-1)!} = -2(-1)^n (C_{2n-2}^n - C_{2n-2}^n) \in \mathbb{Z}. \]

**Problem 21.24.** Let \( u = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}, \quad v = \sum_{n=0}^{\infty} \frac{x^{3n+1}}{(3n+1)!} \) and \( w = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{(3n+2)!}. \) Show that \( u^3 + v^3 + w^3 - 3uvw = 1. \)

**Solution:** It is easy to see that the radius of convergence of each of these power series is \( \infty \). We remark that \( u = u'', \ v = w' \) and \( u + v + w = e^x \). The function \( w \) satisfies then the differential equation \( w + w' + w'' = e^x \), with then initial conditions \( w(0) = 0 \) and \( w'(0) = 0 \). The general solution of the differential equation is
\[ w(x) = \frac{1}{3} e^x + ae^{-x/2} \cos \frac{\sqrt{3}}{2} x + be^{-x/2} \sin \frac{\sqrt{3}}{2} x. \]
Using the initial conditions we get \( a = -\frac{1}{3} \) and \( b = -\frac{1}{\sqrt{3}}. \) So
\[ w = \frac{1}{3} e^x - \frac{1}{3} e^{-x/2} \cos \frac{\sqrt{3}}{2} x - \frac{1}{\sqrt{3}} e^{-x/2} \sin \frac{\sqrt{3}}{2} x \]
\[ v = \frac{1}{3} e^x - \frac{1}{3} e^{-x/2} \cos \frac{\sqrt{3}}{2} x + \frac{1}{\sqrt{3}} e^{-x/2} \sin \frac{\sqrt{3}}{2} x \]
\[ u = \frac{1}{3} e^x + \frac{2}{3} e^{-x/2} \cos \frac{\sqrt{3}}{2} x \]

The identity to prove follows from a simple calculation using
\[ u^3 + v^3 + w^3 - 3uvw = \frac{1}{2} (u + v + w) [(u - v)^2 + (v - w)^2 + (w - u)^2] \]

**Second solution:** Denote \( f = u^3 + v^3 + w^3 - 3uvw \). Then \( f' = 3u^2w + 3v^2u + 3w^2v - 3v^2w - 3u^2w - 3uw^2 \) = 0. The function \( f \) is constant and \( f(x) = f(0) = 1 \) for all \( x \).
Problem 21.25. Suppose that \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) for \( x \in (-R, R) \), where \( R > 0 \).

(a) If \( f \) is an odd function, then \( c_0 = c_2 = c_4 = \ldots = 0 \).
(b) If \( f \) is an even function, then \( c_1 = c_3 = c_5 = \ldots = 0 \).

Solution: (a) The condition \( f(x) = -f(-x) \) writes \( \sum_{n=0}^{\infty} c_{2n} x^{2n} = 0 \) for all \( x \in (-R, R) \), so all the even coefficients must be 0.
(b) Similar to (a).

Problem 21.26. Find the real function whose power series is \( \sum_{n=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!} \).

Solution: Denote \( f(x) = \sum_{n=0}^{\infty} \frac{x^{4k+3}}{(4k+3)!} \). Then \( f(x) \) is solution of the differential equation \( f(x) + f'(x) + f''(x) + f^{(3)}(x) = e^x \), with initial conditions \( f(0) = f'(0) = f''(0) = 0 \). The general solution of this equation is \( f(x) = \frac{1}{4} e^x + a e^{-x} + b e^{ix} + c e^{-ix} \).

From the initial conditions one finds \( a, b, c \).

Problem 21.27. If \( D \) is the differential operator \( x \frac{d}{dx} \), prove that \( e^D P(x) = P(e^x) \), for any polynomial \( P \).

Solution: Because of the linearity of \( D \), it suffices to prove the result for \( P(x) = x^n, n \in \mathbb{N} \). It is easily proved by induction that \( D^n(x^n) = n^n x^n \), for any nonnegative integers \( m, n \). Therefore, \( e^D(x^k) = \left( \sum_{m=0}^{\infty} \frac{D^m}{m!} \right) x^n = \sum_{m=0}^{\infty} \frac{n^m x^n}{m!} = e^n x^n \).

Problem 21.28.
1. Prove that the function \( f: \mathbb{R} \to \mathbb{R} \), defined by \( f(x) = x \) for \( x \leq 0 \) and \( f(x) = \sin \frac{1}{x} + \frac{1}{x} \cos x \) for \( x > 0 \), is not a derivative.
2. Let \( f: \mathbb{R} \to \mathbb{R} \) be a differentiable function with continuous derivative. Prove that the function \( g \) defined by \( g(x) = f(x) \sin \frac{1}{x} \) for \( x \neq 0 \) and \( g(0) = 0 \) is a derivative.

Solution: b) The function defined by \( \phi(x) = x^2 f(x) \cos \frac{1}{x} \) for \( x \neq 0 \), and \( \phi(0) = 0 \) is derivable with \( \phi'(x) = (2xf(x) + x^2 f'(x)) \cos \frac{1}{x} + f(x) \sin \frac{1}{x} \) for \( x \neq 0 \), and \( \phi'(0) = 0 \). The function \( \psi(x) = (2xf(x) + x^2 f'(x)) \cos \frac{1}{x} \), for \( x \neq 0 \) and \( \psi(0) = 0 \), is continuous, hence is a derivative. Then \( g(x) = \phi'(x) - \psi(x) \) is a derivative.

Problem 21.29. Let \( f: [0, 1] \to \mathbb{R} \) be a derivable function with \( f(0) = f(1) = 0 \). Prove there are \( a, b \in (0, 1) \), \( a \neq b \) such that \( |f'(a)| f'(b)| \geq 4 \max_{0 < x < 1} f^2(x) \). As a consequence \( \max |f'(x)| \geq 2 \max |f(x)| \).

Solution: Let \( x_0 \in (0, 1) \) be such that \( |f(x_0)| = \max |f(x)| \). The mean value theorem guarantees the existence of \( a \in (0, x_0) \) and \( b \in (x_0, 1) \) such that \( f(x_0) -
\[ f(0) = x_0f'(a) \text{ and } f(1) - f(x_0) = (1-x_0)f'(b). \] Then \( 4 \max_{0<s<1} f^2(x) = 4|f(x_0)|^2 = 4x_0(1-x_0)|f'(a)f'(b)| \leq |f'(a)f'(b)|. \)

**Problem 21.30.** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable function (with continuous derivative) such that \( f'(x)^2 + f^2(x) > 0 \) for all \( x \in [a, b] \). Prove that \( f \) has only a finite number of zeros.

**Problem 21.31.** Let \( f : [0, 1] \to \mathbb{R} \) be a \( C^1 \)-function, with \( f(0) = 0 \). Prove that
\[
\sup_{0 \leq x \leq 1} |f^2(x)| \leq \int_0^1 (f'(x))^2 \, dx.
\]

**Solution:** For any \( t \in [0, 1] \) we have
\[
1. \int_0^1 (f'(x))^2 \, dx \geq t \int_0^t (f'(x))^2 \, dx = \left( \int_0^t 1^2 \, dx \right) \left( \int_0^t (f'(x))^2 \, dx \right) \geq \left( \int_0^t f'(x) \, dx \right)^2 = f(t)^2.
\]

**Problem 21.32.** Let \( f : \mathbb{R} \to \mathbb{R} \) be a differentiable function such that \( f(x+y) \leq f(x) + f(y) \), for all \( x, y \in \mathbb{R} \). If \( f(0) \leq 0 \) prove there is \( a \in \mathbb{R} \) such that \( f(x) = ax \) for all \( x \in \mathbb{R} \).

**Solution:** From \( f(0+0) \leq f(0)+f(0) \) it follows that \( f(0) = 0 \). For any \( h > 0 \) we have
\[
\frac{f(x+h) - f(x)}{h} \leq \frac{f(h)}{h}
\]
so passing at limit with \( h \to 0 \) we obtain \( f'(x) \leq f'(0) \). Similarly, for \( h < 0 \) we have
\[
\frac{f(x+h) - f(x)}{h} \geq \frac{f(h)}{h}
\]
and passing at limit with \( h \to 0 \) we get \( f'(x) \geq f'(0) \). Hence \( f'(x) \) is constant and \( f \) is linear.

**5. The mean value theorem**

**Problem 21.33.** 

a) Solve in real numbers the equation \( 2^x + 4^x = 3^x + 5^x \).

b) Solve in real numbers the equation \( 2^x + 5^x = 3^x + 4^x \).

**Solution:** 

a) For \( x > 0 \), \( 3^x + 5^x > 2^x + 4^x \), and for \( x < 0 \), \( 3^x + 5^x < 2^x + 4^x \), so \( x = 0 \) is the only solution.

b) The mean value theorem for the function \( f(t) = t^x \) shows there is \( c \in (2, 3) \) and \( d \in (4, 5) \) such that \( 3^x - 2^x = xc^{x-1} \) and \( 5^x - 4^x = xd^{x-1} \). The equation becomes \( x \left( \frac{d}{c} \right)^{x-1} - 1 \) = 0, and has the solutions \( x = 0 \) and \( x = 1 \).

**Problem 21.34.** Solve in real numbers the equation \( 5^x + 5x^2 = 4^x + 6x^2 \).

**Solution:** Write the equation as \( 5^x - 4^x = 6^x + 5x^2 - 4x^2 \). Using the mean theorem for the function \( f(t) = t^x \), there is \( c \in (4, 5) \) such that \( 5^x - 4^x = xc^{x-1} \). Similarly for the function \( g(t) = t^{x^2} \), there is \( d \in (5, 6) \) such that \( 6^x - 5^x = xd^{x-1} \). The
equation becomes \( xc^{x-1} = x^2 d^{x^2-1} \). One obvious solution is \( x = 0 \). Looking for non-zero solutions, we see that necessarily \( x > 0 \).

Suppose \( 0 < x < 1 \). Then \( x = \left( \frac{d^{1+x}}{e} \right)^{1-x} > 1 \). Contradiction.

For \( x > 1 \), we obtain the contradiction \( x = \left( \frac{d^{1+x}}{e} \right)^{1-x} < 1 \). Therefore \( x = 0 \) and \( x = 1 \) are the only solutions.

**Problem 21.35.** Evaluate the limit \( \lim_{n \to \infty} n^2 \left( \left( 1 + \frac{1}{n+1} \right)^{n+1} - \left( 1 + \frac{1}{n} \right)^n \right) \).

**Problem 21.36.** Let \( f(x) = (x^2 - 1)^n \), and let \( P_n(x) \) be the \( n \)-th derivative of \( f(x) \). Prove that \( P_n(x) \) is a polynomial of degree \( n \) with \( n \) real, distinct roots in \((-1, 1)\).
Classes of functions

1. Convex functions

Definition 22.1. A function \( g : I \to \mathbb{R} \) is said to be convex on the interval \( I \) if and only if
\[
g(tx + (1 - t)y) \leq tg(x) + (1 - t)g(y)
\]
for all \( x \) and \( y \) in \( I \) and \( t \in [0, 1] \).

Problem 22.1. Let \( f : I \to \mathbb{R} \) (where \( I \) is an interval of \( \mathbb{R} \)) be such that \( f(x) > 0, x \in I \). Suppose that \( e^{cx}f(x) \) is convex in \( I \) for every real number \( c \). Show that \( \log f(x) \) is convex in \( I \).

Solution: The condition that \( e^{cx}f(x) \) is convex can be written
\[
f(tx + (1 - t)y) \leq te^{c(x-y)(1-t)}f(x) + (1 - t)e^{-ct(x-y)}f(y)
\]
We look at the right side of this inequality as function of \( c \). This function is minimized at \( c = \frac{\ln f(y) - \ln f(x)}{x - y} \). The inequality still holds for this \( c \) and becomes
\[
f(tx + (1 - t)y) \leq f(x)^t f(y)^{1-t}
\]
Taking the logarithm we obtain exactly the convexity of \( \log f(x) \).

Problem 22.2. Let \( f : [a, b] \to \mathbb{R} \) be a convex function. Then \( f(x) \leq \max(f(a), f(b)) \), for any \( x \in [a, b] \).

Problem 22.3. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Then \( f \) is convex if and only if
\[
f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2}, \text{ for all } x, y \in [a, b].
\]

Problem 22.4. Application to inequalities: Jensen’s inequality, the inequality between the means.

Problem 22.5. Let \( f : \mathbb{R} \to \mathbb{R} \) be a derivable function such that for \( a, b \) reals we have
\[
\int_a^{a+b} f(t) \, dt \leq \int_a^b f(t) \, dt
\]
Prove that \( f \) is an increasing function.

Solution: Let \( F \) be an antiderivative of \( f \). Then \( F \) satisfies \( F\left(\frac{a+b}{2}\right) \leq \frac{F(a) + F(b)}{2} \) for all \( a, b \) reals. Since \( F \) is continuous it follows that \( F \) is convex. But \( F \) is twice derivable, so \( F'' = f' \geq 0 \) and as a consequence \( f \) is increasing.
CHAPTER 23

Integrals

1. Antiderivatives

**Problem 23.1.** Prove that the function $f$ defined by $f(x) = \cos \frac{1}{x}$, for $x \neq 0$, and $f(0) = a$, has an antiderivative if and only if $a = 0$.

2. Riemann sums

**Problem 23.2.** Evaluate \( \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n} \sqrt{n+k}} \).

**Solution:** If $f(x) = \frac{1}{\sqrt{1+x}}$, then
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n} \sqrt{n+k}} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1+k/n}} = \int_{0}^{1} f(x) dx = 2\sqrt{2} - 2
\]

**Problem 23.3.** Use an integral to estimate the sum $\sum_{k=1}^{10000} \sqrt{k}$.

**Solution:** For large values of $n$ one can approximate
\[
\sum_{k=1}^{n} \sqrt{k} = n \sqrt{n} \left( \frac{1}{n} \sum_{k=1}^{n} \sqrt{k} \right) \sim n \sqrt{n} \int_{0}^{1} \sqrt{x} dx = 2n \sqrt{n}
\]

**Problem 23.4.** Suppose that $f$ is a real valued function of one real variable such that $\lim_{x \to c} f(x)$ exists for all $c \in [a,b]$. Show that $f$ is Riemann integrable on $[a,b]$.

**Problem 23.5.** Evaluate $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k^2}}$.

**Solution:** $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2 + k^2}} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1 + k^2/n^2}} = \int_{0}^{1} \frac{1}{\sqrt{1 + x^2}} dx = \ln(1 + \sqrt{2})$

**Problem 23.6.** Evaluate $\lim_{n \to \infty} \frac{1}{n^4} \prod_{k=1}^{2n} (n^2 + k^2)^{1/n}$
Solution: Let \( a_n = \frac{1}{n^4} \prod_{k=1}^{2n} (n^2 + k^2)^{1/n} \). Then \( \ln(a_n) = \frac{1}{n} \sum_{k=1}^{2n} \ln \left( 1 + \frac{k^2}{n^2} \right) \rightarrow \int_0^2 \ln(1 + x^2) \, dx. \)

3. The definite integral

**Problem 23.7.** Let \( a, b \) be real numbers, \( b > 0 \), and \( f : [a - b, a + b] \rightarrow \mathbb{R} \) be a continuous function.

a) Prove that if for any \( h \in [-b, b] \) we have

\[
\int_{a-h}^{a+h} f(x) \, dx = 2 \int_{a-h}^{a} f(x) \, dx = 2 \int_{a}^{a+h} f(x) \, dx,
\]

then the graph of \( f \) is symmetrical with respect to the line \( x = a \).

b) If for any \( h \in [-b, b] \) we have

\[
\int_{a-h}^{a+h} f(x) \, dx = 0,
\]

then the graph of \( f \) is symmetrical with respect to the point \((a, f(a))\).

**Problem 23.8.**

(i) Prove that for any positive integer \( n \)

\[
I_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx = \frac{1 \cdot 3 \cdots (2n - 1) \pi}{2 \cdot 4 \cdots (2n)} \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right)
\]

(ii) Prove that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \).

(iii) Prove that \( \lim_{n \to \infty} n^k \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right) \).

**Solution:** (i) Integrate by parts \( I_n = x^2 \cos^{2n-1} x \sin x \bigg|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (2x \cos^{2n-1} x \sin x - (2n - 1)x^2 \cos^{2n-2} x \sin^2 x) \, dx = \frac{1}{n} \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx + (2n - 1)I_{n-1} - (2n - 1)I_n. \)

We calculate separately by parts \( \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \frac{1 \cdot 3 \cdots (2n - 1) \pi}{2 \cdot 4 \cdots (2n)} \) and we obtain the following recurrence

\[
I_n = -\frac{1}{n^2} \frac{\pi^3}{4} \cdot 3 \cdots (2n - 1) + \frac{2n - 1}{2n} I_{n-1}
\]

An easy induction argument ends the proof.

(ii) Consequence of (iii).

(iii) From (i), \( n^k \left( \frac{\pi^2}{6} - \sum_{k=1}^{n} \frac{1}{k^2} \right) = n^k \frac{4 \cdot 2^4 \cdots (2n - 2)}{\pi^3 \cdot 3 \cdots (2n - 1)} \leq 4n^{k+1} I_n. \)

To finish the proof, use the theorem of mean for integrals which assures the existence of \( c \in (0, \frac{\pi}{2}) \) such that \( I_n = \frac{\pi}{2} c^2 \cos^{2n} c \) and \( \lim_{n \to \infty} n^{k+1} \cos^{2n} c = 0. \)
Problem 23.9. Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function. Show that
\[
\lim_{n \to \infty} \frac{1}{n} \int_0^1 f(nx)dx = \int_0^1 xf(x)dx
\]

Solution: We have
\[
0 \leq \left| \frac{1}{n} \int_0^1 f(nx)dx - \int_0^1 xf(x)dx \right| = \left| \sum_{k=1}^{n} \int_{k-1/n}^{k/n} f(x) \left( \frac{nx}{n} - x \right) dx \right| \leq \sum_{k=1}^{n} \int_{k-1/n}^{k/n} |f(x)| dx
\]
\[
\int_{k-1/n}^{k/n} \frac{|f(x)|}{n} dx = \frac{1}{n} \int_0^1 |f(x)| dx \to 0
\]

Problem 23.10. Let \( f \) be a differentiable function defined in the closed interval \([0,1]\) and such that \(|f'(x)| \leq M\) for any \( x \in (0,1) \). Prove that
\[
\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) \right| \leq \frac{M}{n}
\]

Solution: We have
\[
\left| \int_0^1 f(x)dx - \frac{1}{n} \sum_{k=1}^{n} f \left( \frac{k}{n} \right) \right| \leq \sum_{k=1}^{n} \int_{k-1/n}^{k/n} |f(x) - f \left( \frac{k}{n} \right)| dx. \quad \text{It suffices to remark that} \quad |f(x) - f \left( \frac{k}{n} \right)| \leq \frac{1}{n} M, \quad \text{for any} \ x \ \text{in the interval} \ \left( \frac{k-1}{n}, \frac{k}{n} \right)
\]

Problem 23.11. Evaluate the limit
\[
\lim_{n \to \infty} n \left( \ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right)
\]

Solution: Denote \( a_n = n \left( \ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{2n} \right) \). We have
\[
a_n = n \sum_{k=1}^{n} \int_{k-1/n}^{k/n} \left( f(x) - f \left( \frac{k}{n} \right) \right) dx, \quad \text{where} \quad f(x) = \frac{1}{1+x}. \quad \text{To prove the limit is} \ \frac{1}{4} \ \text{we remark that} \quad \left| a_n - \frac{1}{4} \right| \leq I_n + J_n, \quad \text{where}
\]
\[
I_n = \left| a_n - n \sum_{k=1}^{n} \int_{k-1/n}^{k/n} \left( x - \frac{k}{n} \right) f' \left( \frac{k}{n} \right) dx \right|
\]
\[
J_n = \left| n \sum_{k=1}^{n} \int_{k-1/n}^{k/n} \left( x - \frac{k}{n} \right) f' \left( \frac{k}{n} \right) dx - \frac{1}{4} \right|
\]

We evaluate separately \( I_n \leq n \sum_{k=1}^{n} \int_{k-1/n}^{k/n} \left| f(x) - f \left( \frac{k}{n} \right) \right| - \left( x - \frac{k}{n} \right) f' \left( \frac{k}{n} \right) \ dx \).

For any \( x \in \left( \frac{k-1}{n}, \frac{k}{n} \right), \) there is \( c \in \left( \frac{k-1}{n}, \frac{k}{n} \right) \) such that \( \left| f(x) - f \left( \frac{k}{n} \right) \right| - \left( x - \frac{k}{n} \right) f' \left( \frac{k}{n} \right) \)
\[ \left| \frac{1}{2} \left( x - \frac{k}{n} \right)^2 \right| f''(c) = \left| \frac{1 - k}{n} \right| \sqrt{\left( x - \frac{k}{n} \right)^2} \leq \frac{1}{n^2}. \] Therefore, \( I_n \leq \frac{1}{n}. \) On the other side, \( J_n = \frac{1}{2n} \sum_{k=1}^{n} \frac{1}{(1 + \frac{k}{n})^2} - \frac{1}{4} \frac{1}{\sqrt{n}} \left( \frac{1}{\sqrt{1 + 1/n} - 1} \right). \)

**Problem 23.12.** Evaluate \( \lim_{n \to \infty} n \int_{0}^{1} \frac{x^{2n}}{x+1} \, dx. \)

**Solution:**

We have

\[ n \int_{0}^{1} \frac{x^{2n}}{x+1} \, dx = n \int_{0}^{1} \left( x^{2n-1} - x^{2n-2} + \ldots + x - 1 + \frac{1}{x+1} \right) \, dx \]

\[ = n \left( \frac{1}{2n} - \frac{1}{2n-1} + \ldots + \frac{1}{2} - 1 + \ln 2 \right) = n \left( \ln 2 - \frac{1}{n+1} - \frac{1}{n+2} - \ldots - \frac{1}{2n} \right) \]

Using the problems 23.11 or 9.25 we obtain the limit \( \frac{1}{4}. \)

**Problem 23.13.** Let \( f : [0,1] \to \mathbb{R} \) be a \( C^1 \)-function, with \( f(0) = 0. \) Prove that

\[ \sup_{0 \leq x \leq 1} |f^2(x)| \leq \int_{0}^{1} (f'(x))^2 \, dx. \]

**Solution:**

For any \( t \in [0,1], \) using the integral form of Cauchy-Schwarz inequality, we have

\[ \int_{0}^{1} (f'(x))^2 \, dx \geq t \int_{0}^{1} (f'(x))^2 \, dx = \left( \int_{0}^{t} 1^2 \, dx \right) \left( \int_{0}^{1} (f'(x))^2 \, dx \right) \geq \left( \int_{0}^{t} f'(x) \, dx \right)^2 = f(t)^2 \]

**Problem 23.14.** Let \( f \) be a function such that \( f' \) is continuous. Evaluate

\[ \lim_{n \to \infty} \int_{0}^{1} f(x) \sin nx \, dx \]

**Solution:**

An integration by parts gives

\[ \int_{0}^{1} f(x) \sin nx \, dx = \frac{f(0)}{n} - \frac{f(1)\cos n}{n} + \int_{0}^{1} \frac{f'(x)\cos nx}{n} \, dx \]

But \( \left| \int_{0}^{1} \frac{f'(x)\cos nx}{n} \, dx \right| \leq \frac{1}{n} \int_{0}^{1} |f'(x)| \cos nx \, dx \leq \frac{1}{n} \int_{0}^{1} |f'(x)| \, dx. \) Hence \( \lim_{n \to \infty} \int_{0}^{1} f(x) \sin nx \, dx = 0. \)

**Problem 23.15.** Evaluate \( \int \frac{x^2 + 1}{x^4 + 1} \, dx, \int \frac{x^2 - 1}{x^4 + 1} \, dx \) and \( \int \frac{1}{x^4 + 1} \, dx. \)
Solution: With the substitution $y = x - \frac{1}{x}$ respectively $z = x + \frac{1}{x}$ we have

$$\int \frac{x^2 + 1}{x^4 + 1} \, dx = \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} \, dx = \int \frac{dy}{y^2 + 2} = \frac{1}{\sqrt{2}} \arctan \frac{y}{\sqrt{2}} = \frac{1}{\sqrt{2}} \arctan \frac{x^2 - 1}{\sqrt{2}x}$$

$$\int \frac{x^2 - 1}{x^4 + 1} \, dx = \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} \, dx = \int \frac{dz}{z^2 - 2} = \frac{1}{2\sqrt{2}} \ln \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1}$$

For the last integral we substract the preceding two identities.

**Problem 23.16.** If $f$ is an invertible function and $f’$ is continuous, prove that

$$\int_a^b f(x) \, dx = bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(y) \, dy$$

In the case where $f$ and $g$ are positive functions and $b > a > 0$, draw a diagram to give a geometric interpretation of this identity.

**Solution:** An integration by parts gives

$$\int_a^b f(x) \, dx = xf(x)|_a^b - \int_a^b xf’(x) \, dx$$

To finish the proof we notice that the substitution $y = f(x)$ yields

$$\int_a^b xf’(x) \, dx = \int_a^b f^{-1}(f(x)) f’(x) \, dx = \int_{f(a)}^{f(b)} f^{-1}(y) \, dy$$

Due to the fact that the graphs of $f$ and $f^{-1}$ are symmetrical one to the other with respect to the line $y = x$, the reunion of the regions $R_1$ between the graph of $f$ and the $x$ axis, and $R_0$ between the graph of $f^{-1}$ and the $x$-axis is the rectangle determined by the origin, the $x$-axis, the $y$-axis and the point $(a, b)$.

**Problem 23.17.** Evaluate $I = \int_0^1 (\sqrt[3]{1-x^3} - \sqrt[3]{1-x^7}) \, dx$.

**Solution:** The function $f(x) = \sqrt[3]{1-x^7}$ has the inverse $f^{-1}(x) = \sqrt[3]{1-x^3}$. Then (see problem 23.16) $\int_0^1 f(x) \, dx = f(1) - \int_0^1 f^{-1}(x) \, dx = \int f^{-1}(x) \, dx$. So $I = 0$.

**Problem 23.18.** For any number $c$, we let $f_c(x)$ be the smaller of the two numbers $(x-c)^2$ and $(x-c-2)^2$. Then we define $g(c) = \int_0^1 f_c(x) \, dx$. Find the minimum values of $g(c)$ if $c \in [-2, 2]$.

**Solution:** By definition $f_c(x) = \min\{(x-c)^2, (x-c-2)^2 + 4(1+c-x)\}$. We distinguish the cases:

1. If $c \in [-2, -1], 1+c-x \leq -x \leq 0$, for all $x \in [0, 1]$. Hence $f_c(x) = (x-c-2)^2$ and $g(c) = c^2 + 3c + \frac{7}{3}$. The minimum of $g$ on this interval is $g(-3/2) = 1/12$ and the maximum $g(-2) = g(-1) = 1/3$.
2. If \( c \in (-1, 0) \), \( g(c) = \int_0^{1+c} (x - c)^2 \, dx + \int_{1+c}^1 (x - c - 2)^2 \, dx = -c^2 - c + \frac{1}{3} \).

The maximum of \( g \) on this interval is \( g(-1/2) = 7/12 \) and the minimum \( g(-1) = g(0) = 1/3 \).

3. If \( c \in [0, 2] \), \( 1 + c - x \geq 1 - x \geq 0 \) for all \( x \in [0, 1] \) Hence \( f_c(x) = (x - c)^2 \) and \( g(c) = \int_0^1 (x - c)^2 \, dx = c^2 - c + \frac{1}{3} \). The minimum of \( g \) on this interval is \( g(1/2) = 1/12 \) and the maximum is \( g(2) = 7/3 \).

Therefore, the minimum of \( g \) is \( 1/12 \) and the maximum is \( 7/3 \).

**Problem 23.19.** Find \( \lim_{h \to 0} \frac{1}{h} \int_{2+h}^{2+2h} \sqrt{1 + t^3} \, dt \).

**Solution:** By the mean value theorem there is \( c_h \in (2, 2+h) \) such that

\[
\int_{2+h}^{2+2h} \sqrt{1 + t^3} \, dt = h \sqrt{1 + c_h^3}.
\]

Then the limit is \( \sqrt{1 + 2^3} = 3 \).

**Problem 23.20.** Show that \( \int_{\pi/4}^{\pi/2} \frac{\sin x}{x} \, dx \leq \frac{\sqrt{2}}{2} \).

**Solution:** The function \( f(x) = \frac{\sin x}{x} \) is decreasing on the interval \( [\pi/4, \pi/2] \). Then

\[
\int_{\pi/4}^{\pi/2} f(x) \, dx \leq \left( \frac{\pi}{2} - \frac{\pi}{4} \right) f \left( \frac{\pi}{4} \right) = \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}.
\]

**Problem 23.21.** Show that \( \int_0^1 x \arcsin x \, dx < \frac{\pi}{4} \).

**Solution:** Integrating by parts

\[
\int_0^1 x \arcsin x \, dx = \left( \frac{x^2}{2} \arcsin x \right)_0^1 - \int_0^1 \frac{x^2}{2 \sqrt{1-x^2}} \, dx = \frac{\pi}{4} - \int_0^1 \frac{x^2}{2 \sqrt{1-x^2}} \, dx
\]

**Problem 23.22.** Suppose \( f \) is differentiable on \([0, 1]\), \( f(0) = 0 \), \( f(1) = 1 \), \( f'(x) > 0 \), and \( \int_0^1 f(x) \, dx = 1/3 \). Find the value of the integral \( \int_0^1 f^{-1}(y) \, dy \) and give a geometric interpretation of the result.

**Solution:** The function \( f : [0, 1] \to [0, 1] \) is invertible. The substitution \( y = f(x) \) gives

\[
\int_0^1 f^{-1}(y) \, dy = \int_0^1 x f'(x) \, dx = (xf(x))_0^1 - \int_0^1 f(x) \, dx = 1 - 1/3 = 2/3
\]

**Problem 23.23.** Prove that if \( f \) is continuous, then

\[
\int_0^x f(u)(x-u) \, du = \int_0^x \left( \int_0^u f(t) \, dt \right) \, du.
\]
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Solution : The function \( F(x) = \int_0^x f(t)dt \) is differentiable and satisfies \( F'(x) = f(x), \) \( F(0) = 0. \) Then integrating by parts \( \int_0^x \left( \int_0^u f(t)dt \right) du = \int_0^x F(u)du = xF(x) - \int_0^x uf(u)du = \int_0^x f(u)(x-u)du. \)

**Problem 23.24.** Evaluate \( \int_a^b [x]dx, \) where \( a \) and \( b \) are real numbers with \( 0 \leq a < b. \)

**Solution:** Denote \( n = [a] \) and \( m = [b]. \) If \( m > n, \) then

\[
\int_a^b [x]dx = \int_a^{n+1} [x]dx + \sum_{k=1}^{m-n-1} \int_{n+k}^{n+k+1} [x]dx + \int_m^b [x]dx =
\]

\[
= n(n+1-a) + \sum_{k=1}^{m-n-1} (n+k) + m(b-m)
\]

**Problem 23.25.** Show that \( \frac{1}{17} \leq \int_1^2 \frac{dx}{x^4+1} \leq \frac{7}{24}. \)

**Solution:** Consequence of the inequality \( \frac{1}{17} \leq \frac{1}{x^4+1} \leq \frac{1}{x^4}, \) for any \( x \in [1,2]. \)

**Problem 23.26.** Find the interval \( [a,b] \) for which the value of the integral \( \int_a^b (2 + x - x^2)dx \) is a maximum.

**Solution:** The function \( f(x) = 2 + x - x^2 \) has the zeros \(-1 \) and \( 2, \) and is positive only for \( x \in [-1,2]. \) The interpretation of the integral as an area shows that the maximum is attained for \( a = -1 \) and \( b = 2. \) Indeed, in the sum

\[
\int_a^b f(x)dx = \int_{-1}^a f(x)dx + \int_a^2 f(x)dx + \int_2^b f(x)dx
\]

the first and the last term are negatives.

**Problem 23.27.** Suppose \( f \) is continuous on the interval \([-a,a]. \)

a) If \( f \) is even (i.e. \( f(-x) = f(x) \)), then \( \int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx. \) Interpret this equality on a figure.

b) If \( f \) is odd (i.e. \( f(-x) = -f(x) \)), then \( \int_{-a}^a f(x)dx = 0. \) Interpret this equality on a figure.

c) Evaluate \( \int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{x^6 + 1} dx. \)

**Solution:** a)+b) Substitution \( y = -x \) in \( \int_{-a}^0 f(x)dx. \)

\( c) \) The function is odd, therefore the integral is 0.
PROBLEM 23.28. Suppose \( f \) is continuous on \( \mathbb{R} \).

\[
\begin{align*}
a) & \text{ Prove that } \int_{-a}^{b} f(-x) \, dx = \int_{-b}^{-a} f(x) \, dx. \text{ For the case where } f(x) \geq 0, \text{ draw a diagram to interpret this equation geometrically as an equality of areas.} \\
b) & \text{ Prove that } \int_{a}^{b} f(x+c) \, dx = \int_{a+c}^{b+c} f(x) \, dx. \text{ For the case where } f(x) \geq 0, \text{ draw a diagram to interpret this equation geometrically as an equality of areas.}
\end{align*}
\]

**Solution**: Straightforward.

PROBLEM 23.29. For \( \epsilon > 0 \) evaluate the limit
\[
\lim_{x \to \infty} x^{1-\epsilon} \int_{x}^{x+1} \sin(t^2) \, dt
\]

**Solution**: The substitution \( t^2 = u \) gives
\[
\int_{x}^{x+1} \sin(t^2) \, dt = \int_{x^2}^{(x+1)^2} \frac{1}{\sqrt{u}} \sin u \, du.
\]
By the second mean value theorem for integrals, there is a \( c \in (x^2, (x+1)^2) \) such that
\[
\int_{x^2}^{(x+1)^2} \frac{1}{\sqrt{u}} \sin u \, du = \frac{1}{\sqrt{x}} \int_{x}^{c} \sin u \, du + \frac{1}{\sqrt{x+1}} \int_{c}^{(x+1)^2} \sin u \, du.
\]
Then the following inequality shows the limit is 0
\[
\left| x^{1-\epsilon} \int_{x}^{x+1} \sin(t^2) \, dt \right| \leq \frac{x^{1-\epsilon}}{2x} \left| \cos x^2 - \cos c \right| + \frac{x^{1-\epsilon}}{2(x+1)} \left| \cos c - \cos((x+1)^2) \right| \leq 2x^{-\epsilon}
\]

PROBLEM 23.30. Evaluate the limit \( \lim_{y \to 0} \frac{1}{y} \int_{0}^{\pi} \frac{1}{\tan(y \sin x)} \, dx \).

**Solution**: Let \( f(x, y) = \tan(y \sin x) \) and \( g(y) = \int_{0}^{\pi} f(x, y) \). Then \( \lim_{y \to 0} \frac{g(y)}{y} = g'(0) = \int_{0}^{\pi} \frac{\partial f}{\partial y}(x, 0) \, dx \). But \( \frac{\partial f}{\partial y}(x, y) = \frac{\sin x}{\cos^2(y \sin x)} \), so the limit is \( \int_{0}^{\pi} \sin x \, dx = 2 \).

PROBLEM 23.31. Define \( C(\alpha) \) to be the coefficient of \( x^{1992} \) in the power series about \( x = 0 \) of \( (1 + x)^\alpha \). Evaluate
\[
\int_{0}^{1} \left( C(-y - 1) \sum_{k=1}^{1992} \frac{1}{y + k} \right) \, dy.
\]

**Solution**: A simple computation shows \( C(\alpha) = \frac{\alpha(-1) \ldots (\alpha-1991)}{1992!} \). \( C(-1 - y) = \frac{(y + 1)(y + 2) \ldots (y + 1992)}{1992!} \). We have to evaluate then
\[
\frac{1}{1992!} \int_{0}^{1} \sum_{k=1}^{1992} (y + 1)(y + k - 1)(y + k + 1) \ldots (y + 1992) \, dy =
\]
\frac{1}{1992!} \int_0^1 [(y + 1)(y + 2) \cdots (y + 1992)]' \, dy = \frac{1993! - 1992!}{1992!} = 1992

**Problem 23.32.** Show that the function \( F : [0, 1] \to \mathbb{R} \), defined by \( F(t) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - t \cos^2 x}} \) is an increasing function.

**Solution:** One can see easily that \( 1 - t \cos^2 x > 0 \) for any \( t \) and \( x \) so \( F \) is well defined as the integral of a continuous function. Also if \( t_1 \leq t_2 \) then \( \frac{1}{\sqrt{1 - t_1 \cos^2 x}} \leq \frac{1}{\sqrt{1 - t_2 \cos^2 x}} \) for any \( x \), which by integration shows that \( F(t) \) is an increasing function.

**Problem 23.33.** Determine the value of the integral \( I_n = \int_0^\pi \frac{\sin^2 nx}{\sin^2 x} \, dx \).

**Solution:** \( I_{n+1} = \int_0^\pi \left( \frac{\sin nx \cos x + \cos nx \sin x}{\sin^2 x} \right)^2 \, dx = I_n + \int_0^\pi \cos 2nx \, dx + \int_0^\pi \frac{\cos x \sin 2nx}{\sin x} \, dx = I_n + J_n \), where \( J_n = \int_0^\pi \frac{\cos x \sin 2nx}{\sin x} \, dx \). But an easy computation shows that \( J_n - J_{n-1} = \int_0^\pi (\cos 2nx + \cos(2n-2)x) \, dx = 0 \). Therefore \( J_n = J_1 = \pi \), for all \( n \), and \( I_n \) satisfies the recurrence \( I_{n+1} = I_n + \pi \). Then \( I_n = I_1 + (n-1)\pi = n\pi \).

**Problem 23.34.** Evaluate \( \int \frac{dx}{x + x^{10}} \).

**Solution:** We have \( \int \frac{dx}{x + x^{10}} = \int \frac{x^{-10} \, dx}{1 + x^{-9}} = -\frac{1}{9} \ln(1 + x^{-9}) + C \).

**Problem 23.35.** Let \( F(x) \) be a differential function such that \( F'(a-x) = F'(x) \) for all \( x \in [0, a] \). Evaluate \( I = \int_0^a F(x) \, dx \) and give an example of such a function \( F(x) \).

**Solution:** We evaluate first \( J = \int_0^a x F'(x) \, dx \). With the substitution \( y = a - x \), we have \( J = \int_0^a (a-x) F'(x) \, dx = aF(a) - aF(0) - J \), so \( J = \frac{1}{2} a(F(a) - F(0)) \). With integration by parts \( J = aF(a) - \int_0^a F'(x) \, dx \), so \( I = aF(a) - J = \frac{1}{2} a(F(a) + F(0)) \). As example one can take \( F(x) = x(a-x) \), so \( F(x) = ax^2/2 - x^3/3 \).

**Problem 23.36.** Let \( f(x) \) be a continuous function on \([0, a] \), where \( a > 0 \), such that \( f(x)f(a-x) = 1 \). Prove that there are infinitely many such functions, and evaluate the integral \( I = \int_0^a \frac{dx}{1 + f(x)} \).
Solution : For any $c > 0$, the function $f(x) = c^{x-a/2}$ satisfies the condition $f(x)f(a-x) = 1$. The substitution $y = a - x$ yields, $I = \int_{0}^{a} \frac{dy}{1 + f(a-y)} = \int_{0}^{a} \frac{f(x)dx}{1 + f(x)}$. Therefore $I + I = \int_{0}^{a} \frac{dx}{1 + f(x)} + \int_{0}^{a} \frac{f(x)}{1 + f(x)} = a$ and $I = \frac{a}{2}$.

PROBLEM 23.37. Evaluate
$$\int_{2}^{4} \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x) + \sqrt{\ln(x+3)}}} \, dx.$$ [P1987]

Solution : More general, for $a < b < c$ and $f$ a real function we will compute the integral
$$I = \int_{a}^{b} \frac{\sqrt{f(c-x)}}{\sqrt{f(c-x) + \sqrt{f(x+c-a-b)}}} \, dx.$$ The substitution $x = a + b - y$ leads to
$$I = \int_{a}^{b} \frac{\sqrt{f(y+c-a-b)}}{\sqrt{f(y+c-a-b) + \sqrt{f(c-y)}}} \, dy.$$ We denote the last integral with $J$ and we remark that $I + J = b - a$, thus $I = \frac{b-a}{2}$.

PROBLEM 23.38. Evaluate $\int_{0}^{\pi} \frac{x \sin x}{1 + \cos^2 x} \, dx$

Solution : Denote by $I$ the value of the integral. Then with the substitution $x = \pi - y$ we have $I = \pi \int_{0}^{\pi} \frac{\sin x}{1 + \cos^2 x} \, dx - I$. Therefore $I = \frac{\pi^2}{4}$.

PROBLEM 23.39. Let $f(x)$ be a continuous function on $[0, a]$, where $a > 0$, such that $f(x) + f(a-x)$ does not vanish on $[0, a]$. Evaluate the integral $I = \int_{0}^{a} \frac{f(x)}{f(x) + f(a-x)} \, dx$

Solution : Let $J = \int_{0}^{a} \frac{f(a-x)}{f(x) + f(a-x)} \, dx$. The substitution $t = a-x$ shows that $I = J$. But $I + J = a$, so $I = \frac{a}{2}$.

PROBLEM 23.40. Evaluate $\int_{0}^{\pi/2} \frac{dx}{1 + \sqrt{\tan x}}$.

Solution : It seems that we have an improper integral since the function $\tan x$ is not defined for $\frac{\pi}{2}$, but we replacing the tangent function in terms of sinus and cosinus we can arrange conveniently our integral as $I = \int_{0}^{\pi/2} \frac{\sqrt{\cos x \, dx}}{\sqrt{\cos x + \sqrt{\sin x}}}$. Consider also $J = \int_{0}^{\pi/2} \frac{\sqrt{\sin x \, dx}}{\sqrt{\cos x + \sqrt{\sin x}}}$. The substitution $x \to \frac{\pi}{2}$ gives $I = J$ and also we have $I + J = \frac{\pi}{2}$, so $I = \frac{\pi}{4}$. 
Problem 23.41. Let \( I_m = \int_0^{2\pi} \cos(x) \cos(2x) \cdots \cos(mx) \, dx \). For which integers \( m, 1 \leq m \leq 10 \) is \( I_m \neq 0 \)? \[P1985\]

Solution: The answer is \( m \in \{3, 4, 7, 8\} \). Let \( P_m(x) = \cos(x) \cos(2x) \cdots \cos(mx) \).

By induction one can show there are positive numbers \( a_{m,k}, b_{m,k}, c_{m,k}, d_{m,k} \) such that

\[
P_m(x) = \begin{cases} 
4n^2 + 3n \sum_{k=0}^{4n^2 + 3n} a_{m,k} \cos((2k + 1)x) & , \ m = 4n + 1 \\
4n^2 + 5n + 1 \sum_{k=0}^{4n^2 + 5n + 1} b_{m,k} \cos((2k + 1)x) & , \ m = 4n + 2 \\
4n^2 + 3n \sum_{k=0}^{4n^2 + 3n} c_{m,k} \cos(2kx) & , \ m = 4n + 3 \\
4n^2 + n \sum_{k=0}^{4n^2 + n} d_{m,k} \cos(2kx) & , \ m = 4n 
\end{cases}
\]

Hence, for \( m = 4n + 1 \) and \( m = 4n + 2 \), \( I_m = 0 \) since the integral of any of the terms appearing in the sum of \( P_m(x) \) is 0. For \( m = 4n + 3 \) and \( m = 4n \), all the terms in the sum of \( P_m(x) \) have integral 0 except the one corresponding to \( k = 0 \) which has positive integral, so \( I_m \neq 0 \) in these cases.

Problem 23.42. Let \( 0 < a < b \). Evaluate \( l = \lim_{t \to 0} \left\{ \int_0^1 (bx + a(1 - x))^t \, dx \right\}^{1/t} \)

Solution: One can easily evaluate \( \int_0^1 (bx + a(1 - x))^t \, dx = \frac{b^{t+1} - a^{t+1}}{(b-a)(t+1)}. \)

4. Improper integrals

Definition 23.1. The integral \( \int_0^\infty f(x) \, dx \) is:

- convergent if \( \int_0^\infty f(x) \, dx = \lim_{t \to \infty} \int_0^t f(x) \, dx \)
- absolutely convergent if \( \int_0^\infty |f(x)| \, dx \) is convergent.

Proposition 23.2. If the integral \( \int_0^\infty f(x) \, dx \) is absolutely convergent then is convergent.

Proposition 23.3. Let \( f \) be a continuous function on \([0, \infty)\). Then \( \int_0^\infty f(x) \, dx \) is convergent if and only if for any \( \epsilon > 0 \) there is \( B_\epsilon > 0 \) such that \( |\int_s^t f(x) \, dx| < \epsilon \) for \( s, t \geq B_\epsilon \).
23.4. (the second mean theorem for integrals) If \( f \) and \( g \) are continuous and \( f \) is monotonic on \([a, b]\), then there is \( c \in [a, b] \) such that \( \int_a^b f g \, dx = f(a) \int_a^c g \, dx + f(b) \int_c^b g \, dx \).

**Problem 23.43.** Show that for all \( x > 0 \),

\[
0 < \int_0^\infty \frac{\sin t}{\ln(1 + x + t)} \, dt < \frac{2}{\ln(1 + x)}
\]

**Solution:** Integrating by parts

\[
\int_0^\infty \frac{1}{\ln(1 + x + t)} (-\cos t) \, dt = \left. \frac{-\cos t}{\ln(1 + x + t)} \right|_0^\infty - \int_0^\infty \frac{\cos t}{(1 + x + t) \ln^2(1 + x + t)} \, dt
\]

\[
= \frac{1}{\ln(1 + x)} - \int_0^\infty \frac{\cos t}{(1 + x + t) \ln^2(1 + x + t)} \, dt.
\]

It suffices to remark that \( \int_0^\infty \frac{1}{(1 + x + t) \ln^2(1 + x + t)} \, dt = \frac{1}{\ln(1 + x)} \) and \(-1 \leq \cos t \leq 1\).

**Problem 23.44.** Let \( p > 0 \) be a real number and let \( n \geq 0 \) be an integer. Evaluate

\[
u_n(p) = \int_0^\infty e^{-px} \sin x \, dx
\]

**Solution:** Two successive integration by parts with \( dv = e^{-px} \, dx \) lead to \( u_n(p) = \frac{n(n - 1)}{n^2 + p^2} u_{n-2}(p) \). By induction \( u_{2n}(p) = \prod_{k=1}^{n} \frac{2k(2k - 1)}{4k^2 + p^2} u_0(p) \) and \( u_{2n+1}(p) = \prod_{k=1}^{n} \frac{2k(2k + 1)}{(2k + 1)^2 + p^2} u_1(p) \). It remains to see that \( u_0(p) = \frac{1}{p} \) and again by a double integration by parts \( u_1(p) = \frac{1}{1 + p^2} \).

**Problem 23.45.** For what pairs \((a, b)\) of positive real numbers does the improper integral

\[
\int_b^\infty \left( \sqrt{\sqrt{x} + a - \sqrt{x}} - \sqrt{\sqrt{x} - \sqrt{x} - b} \right) \, dx
\]

converge? [P1995]

**Solution:** The integrand is

\[
\sqrt{\frac{a}{\sqrt{x} + a + \sqrt{x}}} - \sqrt{\frac{b}{\sqrt{x} + \sqrt{x} - b}} = \frac{(a - b)\sqrt{\sqrt{x} + a} + \sqrt{x} - b - b\sqrt{x} + a}{\sqrt{\sqrt{x} + a + \sqrt{x}}(\sqrt{x} + \sqrt{x} - b)(\sqrt{b\sqrt{x} + a} + b\sqrt{x} + \sqrt{a\sqrt{x} + a\sqrt{x} - b})}
\]
If \( a \neq b \) then the integrand is similar at infinity with \( \frac{\sqrt{x}}{x^{1/4}} \) so the integral is divergent. For \( a = b \) the integrand is similar at infinity with \( \frac{1}{\sqrt{x}} \) and the integral is convergent.

**Second solution**: The integral converges iff \( a = b \). This proof uses “big-O” notation and the fact that \((1 + x)^{1/2} = 1 + x/2 + O(x^2)\) for \( |x| < 1 \). (Here \( O(x^2) \) means bounded by a constant times \( x^2 \).)

So
\[
\sqrt{x + a} - \sqrt{x} = x^{1/2}(\sqrt{1 + a/x} - 1) = x^{1/2}(1 + a/2x + O(x^{-2})),
\]
hence
\[
\sqrt{x + a} - \sqrt{x} = x^{1/4}(1 + a/4x + O(x^{-2}))
\]
and similarly
\[
\sqrt{x} - \sqrt{x - b} = x^{1/4}(1 + b/4x + O(x^{-2})).
\]
Hence the integral we’re looking at is
\[
\int_b^\infty x^{1/4}((a - b)/4x + O(x^{-2})) \, dx.
\]
The term \( x^{1/4}O(x^{-2}) \) is bounded by a constant times \( x^{-7/4} \), whose integral converges. Thus we only have to decide whether \( x^{-3/4}(a - b)/4 \) converges. But \( x^{-3/4} \) has divergent integral, so we get convergence if and only if \( a = b \) (in which case the integral telescopes anyway).

**Problem 23.46.** Suppose \( f, g : \mathbb{R} \rightarrow \mathbb{R} \) are continuous functions such that the integrals \( \int_{-\infty}^\infty f^2(x) \, dx \) and \( \int_{-\infty}^\infty g^2(x) \, dx \) are convergent. Prove that \( \int_{-\infty}^\infty f(x)g(x) \, dx \) is convergent.

**Solution**: Use Cauchy-Schwarz inequality.

**Problem 23.47.** Study the convergence of the integral \( f(h) = \int_0^\infty e^{-hx} \frac{\sin x}{x} \, dx \)

**Solution** Pentru \( h > 0 \) integrala este absolut convergenta. Intr-adevar
\[
\int_0^\infty |e^{-hx} \frac{\sin x}{x}| \, dx \leq \int_0^\infty e^{-hx} \, dx = \left(-\frac{1}{h} e^{-hx}\right)|_0^\infty = \frac{1}{h}.
\]

Fie acum \( h = 0 \). Demonstrăm ca integrala \( \int_0^\infty \frac{\sin x}{x} \, dx \) este convergenta dar nu este absolut convergenta. Pentru prima parte folosim criteriul R3. Pentru \( \epsilon > 0 \) fixat luam \( B_\epsilon = \frac{4}{\epsilon} \). Fie \( t \geq s \geq B_\epsilon \). Conform R4 exista \( c \in (s, t) \) astfel incat
\[
|\int_s^t \sin x \, dx| = \left| \frac{1}{s} \int_s^c \sin x \, dx + \frac{1}{t} \int_c^t \sin x \, dx \right| \leq \left| \frac{\cos s - \cos c}{s} \right| + \left| \frac{\cos c - \cos t}{t} \right| \leq \frac{1}{s} + \frac{1}{t} \leq \frac{2}{s} + \frac{2}{t} \leq \frac{2}{B_\epsilon} + \frac{2}{B_\epsilon} = \frac{4}{B_\epsilon} = \epsilon.
\]

Faptul că integrala nu este absolut convergentă reiese din
\[
\int_0^\infty |\frac{\sin x}{x}| \, dx \geq \sum_{n \geq 1} \int_{(n-1)\pi}^{n\pi} |\frac{\sin x}{x}| \, dx \geq \sum_{n \geq 1} \int_{(n-1)\pi}^{n\pi} |\frac{\sin x}{n\pi}| \, dx = \sum_{n \geq 1} \frac{2}{n\pi} = \infty
\]
Problem 23.48. Evaluate \( \int_0^\infty \frac{\arctan(\pi x)}{x} \, dx \).

Solution: We have
\[
\int_0^\infty \frac{\arctan(\pi x)}{x} \, dx \geq \int_{1/\pi}^\infty \frac{\arctan(\pi x)}{x} \, dx \geq \frac{\pi}{4} \int_{1/\pi}^\infty \frac{1}{x} \, dx = \infty.
\]

Problem 23.49. Find the value of the constant \( C \) for which the integral
\[
\int_0^\infty \left( \frac{1}{\sqrt{x^2 + 4}} - \frac{C}{x + 2} \right) \, dx
\]
converges.

Solution: The integral is convergent if and only if \( C = 1 \). Indeed, from
\[
\int_0^\infty \left( \frac{1}{\sqrt{x^2 + 4}} - \frac{C}{x + 2} \right) \, dx = \int_0^\infty \frac{x + 2 - C\sqrt{x^2 + 4}}{(x + 2)\sqrt{x^2 + 4}} \, dx
\]
we see that necessarily \( C = 1 \), otherwise the fraction is of the order of \( x \). For \( C = 1 \) the integral is
\[
\int_0^\infty \left( \frac{1}{\sqrt{x^2 + 4}} - \frac{1}{x + 2} \right) \, dx = \ln \frac{x + \sqrt{x^2 + 4}}{x + 2} \bigg|_0^\infty = \ln \frac{6}{1 + \sqrt{5}}.
\]

Problem 23.50. Evaluate \( I = \int_0^\infty e^{-x^2} \, dx \).

Solution: \( I^2 = \left( \int_0^\infty e^{-x^2} \, dx \right) \left( \int_0^\infty e^{-y^2} \, dy \right) = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \, dxdy \). With the change of variables \( x = r \cos t, y = r \sin t \), we have \( I^2 = \int_0^{\pi/2} \int_0^\infty re^{-r^2} dt \, dr = \pi \). Therefore \( I = \sqrt{\pi} \).

Problem 23.51. Evaluate \( J = \int_0^{\pi/2} \ln(\sin x) \, dx \).

Solution: First remark the integral is convergent, since \( \lim_{x \to 0} \frac{\ln(\sin x)}{\ln x} = 1 \) and \( \int_0^{\pi/2} \ln x \, dx \) is convergent. Next using \( J = \int_0^{\pi/2} \ln(\cos x) \, dx \) we have
\[
2J = \int_0^{\pi/2} \ln(\sin 2x) \, dx = \frac{\pi}{2} \ln 2 = \int_0^{\pi/2} \frac{1}{2} \ln(\sin y) \, dy - \frac{\pi}{2} \ln 2 = J - \frac{\pi}{2} \ln 2.
\]
Hence \( J = -\frac{\pi}{2} \ln 2 \).

Problem 23.52. Find all positive \( a \) such that \( \int_0^a x^{-\ln x} \, dx = \int_a^{\infty} x^{-\ln x} \, dx \). Evaluate the integrals for these values of \( a \).

Solution: With the change of variables \( \ln x = y \) the equation becomes \( \int_{-\infty}^{\ln a} e^{y-y^2} \, dy = \int_{\ln a}^{\infty} e^{y-y^2} \, dy \). We change again the variable \( t = y - \frac{1}{2} \) and write the equation as
f(ln a − 1/2) = f(∞) − f(ln a − 1/2), where \( f(u) = \int_{-\infty}^{u} e^{-t^2} dt \). Using the problem 23.50, the equation becomes \( f(ln a − 1/2) = \sqrt{\pi} = f(0) \). But \( f \) is an increasing function, so \( a = e^{1/2} \), and the original integrals are then \( e^{1/4} \sqrt{\pi} \).

**Problem 23.53.** Evaluate \( \int_{0}^{\infty} \left( x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots \right) \left( 1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots \right) dx \).

**Solution:** We remark that \( x - \frac{x^3}{2} + \frac{x^5}{2 \cdot 4} - \frac{x^7}{2 \cdot 4 \cdot 6} + \cdots = x \sum_{k=0}^{\infty} \frac{(-x^2/2)^k}{k!} = xe^{-x^2/2} \), so we have to evaluate \( A = \int_{0}^{\infty} xe^{-x^2/2} \sum_{k=0}^{\infty} \frac{(x^4)^k}{(k!)^2} dx = \sum_{k=0}^{\infty} \frac{(1/2)^k}{(k!)^2} I_k \),

where \( I_k = \int_{0}^{\infty} xe^{-x^2/2} \left( \frac{x^2}{2} \right)^k dx \). The substitution \( y = x^2/2 \) followed by an integration by parts shows that \( I_k = kI_{k-1} \), which leads to \( I_k = k! \). Therefore \( A = e^{1/2} \).

**Problem 23.54.** Given that \( \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \), evaluate \( \int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx \).

**Solution:** We integrate by parts \( \int_{0}^{\infty} \frac{\sin^2 x}{x^2} dx = - \int_{0}^{\infty} (\sin^2 x)' \left( \frac{1}{x} \right) dx = \int_{0}^{\infty} \frac{\sin 2x}{x} dx = \frac{\pi}{2} \)

5. Applications of the integral

**Problem 23.55.** The base of a solid is a square with vertices located at \((1,0)\), \((0,1)\), \((-1,0)\) and \((0,-1)\). Each cross section perpendicular to the x-axis is a semi-circle. Find the volume of the solid.

**Solution:** The area of the section perpendicular to the x-axis at the point \( x \) is a semi-circle with radius \(|x|\), and area \( A(x) = \frac{\pi x^2}{2} \). The volume of the solid is therefore \( V = \int_{-1}^{1} A(x) dx = \frac{\pi}{3} \).

Another solution is based on the idea that cutting the solid along the y-axis one can glue the two parts into a cone with radius 1 and height 1.

**Problem 23.56.** By comparing areas, show that \( \ln 2 < 1 < \ln 3 \). Deduce that \( 2 < e < 3 \).

**Solution:** We have \( \ln 2 = \int_{1}^{2} \frac{1}{x} dx < (2 - 1) \frac{1}{1} = 1 \). On the other side the region situated under the graph of the function \( f(x) = \frac{1}{x} \), between the points \((1,1)\) and
(3, 1/3) contains the triangle with vertices (1, 1), (1, 0) and (3, 1/3). The area of this triangle is 1, therefore \( \ln 3 = \int_{1}^{3} \frac{1}{x} \, dx > 1 \).

**Problem 23.57.** By comparing areas, show that
\[
\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \ln n < \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}
\]

**Solution:** We have \( \ln n = \int_{1}^{n} \frac{1}{x} \, dx = \sum_{k=1}^{n} \int_{k}^{k+1} \frac{1}{x} \, dx \). But comparing areas we have \( \frac{1}{k+1} < \int_{k}^{k+1} \frac{1}{x} \, dx < \frac{1}{k} \).

**Problem 23.58.** Let the function \( f \) from \([0, 1]\) to \([0, 1]\) have the following properties:
- \( f \) is of class \( C^1 \)
- \( f(0) = f(1) = 0 \)
- \( f \) is nonincreasing (i.e. \( f \) is concave)

Prove that the arclength of the graph of \( f \) does not exceed 3.

**Solution:** If the derivative \( f' \) has constant sign, then \( f \) must be monotonic, so must be constant since \( f(0) = f(1) \). The arclength of \( f \) is then 1.

If \( f' \) doesn’t have constant sign, then there is an unique point \( x_0 \in (0, 1) \) such that \( f'(x_0) = 0 \). A majoration of the arclength is given then by \( L(f) = \int_{0}^{1} \sqrt{1 + (f'(x))^2} \, dx \leq \int_{0}^{1} (1 + |f'(x)|) \, dx = 1 + \int_{0}^{x_0} f'(x) \, dx - \int_{x_0}^{1} f'(x) \, dx = 1 + 2f(x_0) \leq 3 \).

**Problem 23.59.** Can an arc of a parabola inside a circle of radius 1 have a length greater than 4? [P2001]

**Solution:** The answer is yes. We consider the circle \( x^2 + y^2 = 1 \) and the parabola \( y = -1 + ax^2 \). For \( a \geq \frac{1}{2} \) the circle and the parabola intersect in the points of coordinates \((0, -1), \left(\sqrt{\frac{2a - 1}{a}}, 1 - \frac{1}{a}\right)\) and \(\left(-\sqrt{\frac{2a - 1}{a}}, 1 - \frac{1}{a}\right)\). For symmetry reasons the length of the arc of the parabola inside the circle is
\[
L = 2 \int_{0}^{\sqrt{\frac{2a - 1}{a}}} \sqrt{1 + 4a^2 x^2} \, dx = 4 + 4 \left(\sqrt{1 + t^2} + \ln(t + \sqrt{1 + t^2}) - t^2 - 4\right)
\]
where \( t = 2\sqrt{2a - 1} \in [0, \infty) \). It suffices to show that the function \( g(t) = t\sqrt{1 + t^2} + \ln(t + \sqrt{1 + t^2}) - t^2 - 4 \) can take positive values. But this is straightforward with the remark \( g(t) \geq \ln(t + \sqrt{1 + t^2}) - 4 \).

**Problem 23.60.** The horizontal line \( y = c \) intersects the curve \( y = 2x - 3x^3 \) in the first quadrant as in the figure. Find \( c \) so that the areas of the two shaded regions are equal. [Figure not included. The first region is bounded by the y-axis, the line \( y = c \) and the curve; the other lies under the curve and above the line \( y = c \) between their two points of intersection.] [P1993]
Solution: The two regions in question have the same area if and only if the region $R_1$ which lies under the curve and above the $x$-axis has the same area like the region $R_2$ in the first quadrant bounded by the two axis, the line $y = c$ and the curve. It is easy to compute $\text{Area}(R_1) = \int_0^{\sqrt{2/3}} (2x - 3x^3) \, dx = \frac{1}{3}$. If $(x_0, c)$ the point of intersection of the curve with the line $y = c$, then $\text{Area}(R_2) = cx_0 + \int_{x_0}^{\sqrt{2/3}} (2x - 3x^3) \, dx = cx_0 + \frac{1}{3} - x_0^2 + \frac{3}{4} x_0^4$. The equality of areas is equivalent with $x_0 - \frac{3}{4} x_0^3 = c$. But $c = 2x_0 - 3x_0^3$, so $x_0 = \frac{2}{3}$ as the unique positive solution of $x_0 - \frac{3}{4} x_0^3 = 2x_0 - 3x_0^3$ and consequently $c = \frac{4}{9}$.

Problem 23.61. Find the positive value of $m$ such that the area in the first quadrant enclosed by the ellipse $\frac{x^2}{9} + y^2 = 1$, the $x$-axis, and the line $y = \frac{2x}{3}$ is equal to the area in the first quadrant enclosed by the ellipse $\frac{x^2}{9} + y^2 = 1$, the $y$-axis, and the line $y = mx$. [P1994]

Solution: The intersection point between the ellipse and the line $y = mx$ is $\left( \frac{3}{\sqrt{9m^2 + 1}}, \frac{3m}{\sqrt{9m^2 + 1}} \right)$. Then the area of the region in the first quadrant bounded by the line $y = mx$, the ellipse and the $x$-axis is

$$A_m = \frac{1}{2} \int_{\frac{3}{\sqrt{9m^2 + 1}}}^{\sqrt{1 - \frac{x^2}{9}}} 1 - \frac{x^2}{9} \, dx = \frac{3\pi}{4} - \frac{3}{2} \arcsin \frac{1}{\sqrt{9m^2 + 1}}.$$

Since the area of the ellipse is $3\pi$, the area in the first quadrant enclosed by the ellipse, the $y$-axis and the line $y = mx$ is $\frac{3\pi}{4} - A_m = \frac{3}{2} \arcsin \frac{1}{\sqrt{9m^2 + 1}}$. The required $m$ is then the solution of the equation

$$\frac{3}{2} \arcsin \frac{1}{\sqrt{9m^2 + 1}} = A_{2/3}$$

or equivalently $\arcsin \frac{1}{\sqrt{9m^2 + 1}} = \frac{\pi}{2} - \arcsin \frac{1}{\sqrt{5}}$. We apply the function sin to this equation and we get $9m^2 = \frac{1}{4}$ with the unique positive solution $m = \frac{1}{6}$.

Problem 23.62. Prove or disprove: There is a region $R$ in the plane which has infinite area and such that the volume of the solid $S$ obtained by rotating $R$ about the $x$-axis is finite, and the surface area of $S$ is infinite.

Solution: Consider the curve $y = \frac{1}{x}$ for $x \geq 1$ and $R$ the region between the graph of this curve and the $x$-axis. The area of $R$ is $\int_1^\infty \frac{1}{x} \, dx = \infty$ and the volume of $S$ is $\int_1^\infty \pi \frac{1}{x^2} \, dx = \pi$. Also the surface area is given by $SA = \int_1^\infty 2\pi \frac{1}{x} \sqrt{1 + \left( \frac{1}{x^2} \right) } \, dx \geq \int_1^\infty 2\pi \frac{1}{x} \, dx = \infty$.
Problem 23.63. Show that the surface area of a zone of the sphere that lies between two parallel planes is $S = \pi dh$, where $d$ is the diameter and $h$ the distance between the planes.

Solution: Such a zone of a sphere is obtained rotating the region in the plane which lies between the graph of the function $y = \sqrt{r^2 - x^2}$, the $x$-axis and the vertical lines $x = a$ and $x = a + d$. The surface area of the zone is then $S = \int_a^{a+d} 2\pi \sqrt{r^2 - x^2} \sqrt{1 + \frac{x^2}{r^2 - x^2}} \, dx = 2\pi rh$.

Problem 23.64. For what values of $m$ do the line $y = mx$ and the curve $y = f(x) = \frac{x}{x^2 + 1}$ enclose a region? Find the area of the region.

Solution: Sketching the graph of $y = f(x)$ we see that the necessary and sufficient condition is that $0 < m < f'(0) = 1$. Another way of proving this is to remark that it suffices that the two curves intersect in two points excepting the origin. The equation $mx = \frac{x}{x^2 + 1}$ has a non-zero solution if and only if $m \in (0, 1)$ and in this case $x_1 = -\sqrt{\frac{1 - m}{m}}$, $x_2 = \sqrt{\frac{1 - m}{m}}$. For symmetry reasons the area is the double of the integral $\int_0^{x_2} \left( \frac{x}{x^2 + 1} - mx \right) \, dx = -\frac{1}{2} \ln m + \frac{m - 1}{2}$.

Problem 23.65. Find the area of the region consisting of all points inside a square that are closer to the center than to the sides of the square.

Solution: Suppose the square is determined by the lines $y = 0$, $x = 1$, $x = -1$, $y = 2$. The center of the square has the coordinates $(1, 0)$. We restrict our attention to the region determined by the lines $y = 0$, $x = 0$, $y = 1 - x$. The set of points which have the same distance to the center $(1, 0)$ and the side $y = 0$ is given by the equation $x^2 + (y - 1)^2 = y^2$ or $y = \frac{x^2 + 1}{2}$. The intersection of this parabola and the line $y = 1 - x$ is $(\sqrt{2} - 1, 2 - \sqrt{2})$. Then the area of the region we consider is $8 \int_0^{\sqrt{2}-1} \left( 1 - x - \frac{1 + x^2}{2} \right) \, dx = \frac{4(4\sqrt{2} - 5)}{3}$.

Problem 23.66. (Theorem of Pappus) Let $R$ be a region that lies entirely on one side of a line $l$ in the plane. If $R$ is rotated about $l$, then the volume of the resulting solid is the product of the area $A$ of $R$ and the distance $d$ traveled by the centroid of $R$.

Solution: We give the proof in the special case when the region lies between $y = f(x)$ and $y = g(x)$ and the line $l$ is the $y$-axis. Using the method of cylindrical shells, we have $V = \int_a^b 2\pi x (f(x) - g(x)) \, dx = 2\pi \bar{x} A$.

Problem 23.67. Suppose that a region $R$ has the area $A$ and lies above the $x$-axis. When $R$ is rotated about the $x$-axis, it sweeps out a solid with volume $V_1$. 
When $\mathcal{R}$ is rotated about the line $y = -k$ (where $k$ is a positive number), it sweeps out a solid of volume $V_2$. Express $V_2$ in terms of $V_1, k$ and $A$.

**Solution**: Let $d$ be the distance from the centroid of $\mathcal{R}$ to the $x$-axis. Then, by the theorem of Pappus, $V_1 = d \cdot A$ and $V_2 = (d + k) \cdot A$, so $V_2 = V_1 + k \cdot A$.

**Problem 23.68.** A torus is formed by rotating a circle of radius $r$ about a line in the plane of the circle that is a distance $R > r$ from the center of the circle. Find the volume and the surface area of the torus.

**Solution**: The circle has area $\pi r^2$ and by the symmetry principle, its centroid is in its center and so the distance travelled by the centroid during a rotation is $2\pi R$. Therefore, by the theorem of Pappus, the volume of the torus is $V = (2\pi R)(\pi r^2) = 2\pi^2 r^2 R$.

**Second solution** We obtain the torus rotating the circle $x^2 + (y - R)^2 = r^2$ about the $x$-axis. The volume is given by the integral $V = \int_{-r}^{r} \pi(y_1^2 - y_2^2)dx$, where $y_1 = R + \sqrt{r^2 - x^2}$ and $y_2 = R - \sqrt{r^2 - x^2}$, so $V = 4\pi R \int_{-r}^{r} \sqrt{r^2 - x^2}dx = 2\pi^2 R r^2$.

The surface area is given by $S = \int_{-r}^{r} 2\pi y_1 \sqrt{1 + (y'_1)^2}dx + \int_{-r}^{r} 2\pi y_2 \sqrt{1 + (y'_2)^2}dx$.

But $(y'_1)^2 = (y'_2)^2 = \frac{x^2}{r^2 - x^2}$, so

$$S = \int_{-r}^{r} 2\pi (y_1 + y_2) \frac{r}{\sqrt{r^2 - x^2}}dx = \int_{-r}^{r} 4\pi Rr \frac{1}{\sqrt{r^2 - x^2}}dx = 4\pi^2 R r$$

**Problem 23.69.** Find the volume of the solid obtained by rotating the triangle with vertices $(2, 3)$, $(2, 5)$ and $(5, 4)$ about the $x$-axis.

**Solution**: The centroid of the triangle is $\left(\frac{2 + 2 + 5}{3}, \frac{3 + 5 + 4}{3}\right) = (3, 4)$. The area of the triangle is $3$ and the distance travelled by the centroid during a rotation is $2\pi \cdot 4$. Therefore, by the theorem of Pappus the volume of the solid is $V = 24\pi$.

**Problem 23.70.** Use the theorem of Pappus to find the centroid of the semicircular region bounded by the curve $y = \sqrt{r^2 - x^2}$ and the $x$-axis.

**Solution**: Using the symmetry principle we see that $\bar{x} = 0$. Rotating the semicircular region we get a sphere with the volume $V = \frac{4}{3} \pi r^3$. The area of the semicircular region is $A = \frac{\pi r^2}{2}$. By the theorem of Pappus, $V = A\bar{y}$, so $\bar{y} = \frac{4r}{3\pi}$.

**Problem 23.71.** If the tangent at a point $P$ on the curve $y = x^3$ intersects the curve again at $Q$, let $A$ be the area of the region bounded by the curve and the line segment $PQ$. Let $B$ be the area of the region defined in the same way starting with $Q$ instead of $P$. What is the relationship between $A$ and $B$?

**Solution**: Let the point $P$ be of coordinates $(a, a^3)$. Then the tangent at $P$ has the equation $y = 3a^2 x - 2a^3$, and this tangent intersects the curve again in the
point with abscises given by the solutions of the equation \( x^3 = 3a^2x - 2a^3 \). This equation has the double solution \( x = a \) and also the solution \( x = -2a \). Therefore \( Q \) has the coordinates \((-2a, -8a^3)\). Because of the symmetry with respect to the origin, we can suppose without any restriction of the generality that \( a > 0 \). The area \( A \) is given by

\[
A(a) = \int_{-2a}^{2a} a(x^3 - 3a^2x + 2a^3)dx = \frac{27a^4}{4}
\]

But \( B = A(-2a) \), so \( B = 16A \).

**Problem 23.72.** Let \( R \) be the region that lies between the curves \( y = x^n \) and \( y = x^m \), where \( m \) and \( n \) are integers with \( 0 \leq n < m \). Find values of \( m \) and \( n \) such that the centroid of \( R \) lies outside \( R \).

**Solution:** The intersection points of the two curves are \((0, 0)\) and \((1, 1)\). The area of the region is

\[
A = \int_0^1 (x^n - x^m)dx = \frac{m-n}{(n+1)(m+1)}
\]

The coordinates of the centroid are given by

\[
\bar{x} = \frac{1}{A} \int_0^1 x(x^n - x^m)dx = \frac{(n+1)(m+1)}{(n+2)(m+2)},
\]

\[
\bar{y} = \frac{1}{2A} \int_0^1 (x^{2n} - x^{2m})dx = \frac{(n+1)(m+1)}{(2n+1)(2m+1)}.
\]

It suffices to find values of \( m, n \) such that \( \bar{y} < \bar{x}^n \) or \( \bar{y} > \bar{x}^n \).

**6. Integrals in \( \mathbb{R}^n \)**

**Problem 23.73.** Evaluate

\[
\lim_{n \to \infty} \int_0^1 \int_0^1 \ldots \int_0^1 \cos^2 \left( \frac{\pi}{2n} (x_1 + x_2 + \ldots + x_n) \right) dx_1 dx_2 \ldots dx_n
\]

**Solution:** The substitutions \( x_k = 1 - y_k \) give

\[
\int_0^1 \int_0^1 \ldots \int_0^1 \cos^2 \left( \frac{\pi}{2n} (x_1 + x_2 + \ldots + x_n) \right) dx_1 dx_2 \ldots dx_n = \int_0^1 \int_0^1 \ldots \int_0^1 \sin^2 \left( \frac{\pi}{2n} (y_1 + y_2 + \ldots + y_n) \right) dy_1 dy_2 \ldots dy_n.
\]

But the sum of these two integrals is 1, so the limit is 1/2.

**Problem 23.74.** Evaluate \( \int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} dy \, dx \) where \( a \) and \( b \) are positive. \([P1989]\)

**Solution:** With the substitutions \( x = au \) and \( y = bv \) the integral becomes

\[
I = ab \int_0^1 \int_0^1 e^{a^2b^2 \max\{u^2, v^2\}} dvdu.
\]

Let \( D_1 = \{(u, v) \in [0,1]^2, u \leq v\} \) and \( D_2 = \)
\{ (u, v) \in [0, 1]^2, u \geq v \}. Then

\[ I = ab \int_{D_1} e^{a^2 b^2 v^2} du dv + ab \int_{D_2} e^{a^2 b^2 u^2} du dv \]

\[ = ab \int_0^1 \left( \int_0^v e^{a^2 b^2 v^2} du \right) dv + \int_0^1 \left( \int_0^u e^{a^2 b^2 u^2} dv \right) du = 2ab \int_0^1 t e^{a^2 b^2 t^2} dt = \frac{e - 1}{ab} \]

**Problem 23.75.** Let \( A \in M_n(\mathbb{R}) \) be a positively defined symmetric matrix. Show that

\[ \int_{\mathbb{R}^n} \exp(-X^tAX) dx_1 dx_2 \ldots dx_n = \frac{\pi^{\frac{n}{2}}}{\sqrt{\det A}} \]

where \( X \) denotes the column vector with coefficients \( x_1, x_2, \ldots, x_n \).

**Problem 23.76.** Evaluate \( \int_0^1 \int_0^1 \frac{dx dy}{1 - xy} \).