Fall 2006 Math 283 Exam 1 Section 1 Version A

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If you cannot complete a problem (perhaps because you forgot a formula) but you think you know how, please describe. Correct methods will receive partial credits.

1. Determine whether each statement is true or false (2 points for correct answer, 1 for no answer, and 0 for wrong answer). An extra point each for counterexamples or for correcting false statements.

(a) The traces of \( f(x, y) = x^2 - y^2 \) in \( y = k \) are hyperbolas.

(b) If \( \mathbf{a} \times \mathbf{b} = 0 \) then \( \mathbf{a} = 0 \) or \( \mathbf{b} = 0 \).

(c) The cross product of two unit vectors is a unit vector.

(d) The surface described in the cylindrical coordinates by \( r = 3 \) is a cylinder.

2. We find an equation of the plane through the point \((1, -1, 1)\) containing the line \( x = 4 - t, \ y = 3 + 5t, \ z = 2 - 3t \).

(a) Find two vectors on the plane.

(b) Now find a normal vector \( \mathbf{n} \) for the plane, which can be taken as the cross product of the two vectors you found in part (a) (If you could not do part (a), use \( (1, 2, 3) \) and \( (4, -1, 2) \)).

(c) Finally write an equation for the plane using the vector found in part (b) (if you could not do part (b), use the (incorrect) normal vector \( (-5, 3, 6) \)).
3. The conversion equations from spherical to rectangular coordinates are given by

\[ x = \rho \sin \phi \cos \theta \]
\[ y = \rho \sin \phi \sin \theta \]
\[ z = \rho \cos \phi, \text{ along with} \]
\[ \rho^2 = x^2 + y^2 + z^2. \]

The spherical coordinates of a point are \((5, 3\pi/4, \pi/6)\). Find

(a) the rectangular coordinates of the point.

\[
x = 5 \sin \left( \frac{3\pi}{4} \right) \cos \left( \frac{\pi}{6} \right) = 2.5 \text{ \(\sqrt{2}\)}.
\]
\[
y = 5 \sin \left( \frac{3\pi}{4} \right) \sin \left( \frac{\pi}{6} \right) = 2.5 \text{ \(\sqrt{2}\)}.
\]
\[
z = 5 \cos \left( \frac{3\pi}{4} \right) = 2.5 \text{ \(\sqrt{2}\)}.
\]
\[
(2.5 \text{ \(\sqrt{2}\)}, 2.5 \text{ \(\sqrt{2}\)}, 2.5 \text{ \(\sqrt{2}\)})
\]

(b) the cylindrical coordinates of the point.

\[
\rho = \sqrt{x^2 + y^2} = \sqrt{2.5^2 + 2.5^2} = 3.5.
\]
\[
\theta = \text{the same as in spherical, so } \frac{3\pi}{4}.
\]
\[
\z = \text{as in rectangular, so } 2.5 \text{ \(\sqrt{2}\)}.
\]

4. Find the angle between the vectors \(\mathbf{a} = \langle 1, -1, 2 \rangle\) and \(\mathbf{b} = \langle 0, 1, 3 \rangle\).

\[
\|\mathbf{a}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6},
\]
\[
\|\mathbf{b}\| = \sqrt{0^2 + 1^2 + 3^2} = \sqrt{10},
\]
\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = \frac{5}{\sqrt{6} \cdot \sqrt{10}} = \frac{5}{6 \sqrt{10}}.
\]
\[
\theta = \cos^{-1} \left( \frac{5}{6 \sqrt{10}} \right) \approx 0.87 \text{ or } 50^\circ.
\]

5. Find \(\mathbf{r}(t)\) if \(\mathbf{r}'(t) = 2t \mathbf{i} + 3t^2 \mathbf{j} + 4tk\) and \(\mathbf{r}(1) = \mathbf{i} + \mathbf{k}\).

\[
\mathbf{r}'(t) = 2t \mathbf{i} + 3t^2 \mathbf{j} + 4tk
\]
\[
\mathbf{r}'(1) = 2 \mathbf{i} + 3 \mathbf{j} + 4k.
\]
\[
\mathbf{r}(t) = \mathbf{r}(1) + \int_1^t \mathbf{r}'(t) \, dt = (t^2 + C_1) \mathbf{i} + (t^3 + C_2) \mathbf{j} + (2t^2 + C_3) \mathbf{k}.
\]
\[
\mathbf{r}(1) = (1 + C_1) \mathbf{i} + (1 + C_2) \mathbf{j} + (2 + C_3) \mathbf{k} = \mathbf{i} + \mathbf{k}.
\]
\[
1 + C_1 = 0, \quad C_2 = -1, \quad 2 + C_3 = 1.
\]
\[
\mathbf{r}(t) = t^2 \mathbf{i} + (t^3 - 1) \mathbf{j} + (2t^2 - 1) \mathbf{k}.
\]
6. Shown above is the curve with the vector equation \( \mathbf{r}(t) = (t + 1, \sqrt{t}) \) for \( 0 \leq t \leq 2 \). Answer the following questions.

(a) Find \( \mathbf{r}(1) \).
\[
\mathbf{r}(1) = \langle 2, \sqrt{1} \rangle = \langle 2, 1 \rangle.
\]

(b) Find \( \mathbf{r}'(1) \), the derivative of \( \mathbf{r} \) at \( t = 1 \). Draw the vector in the above figure with its tail at the tip of \( \mathbf{r}(1) \).
\[
\mathbf{r}'(t) = \langle 1, \frac{1}{2} t^{-\frac{1}{2}} \rangle \quad \text{so} \quad \mathbf{r}'(1) = \langle 1, \frac{1}{2} \rangle.
\]

(c) Find parametric equations for the tangent line to the curve \( \mathbf{r}(t) \) at \( t = 1 \).
\[
\mathbf{T} = \langle 2, 1 \rangle + s \langle 1, \frac{1}{2} \rangle.
\]

(d) Is \( \mathbf{r}(t) \) smooth for \( 0 < t \leq 2 \)? (That is, is the derivative \( \mathbf{r}'(t) \) defined for each \( t \) and \( \mathbf{r}'(t) \neq 0 \)?)
\[
\text{Yes}
\]

7. Find a vector function that represents the curve of intersection of the two surfaces \( z = \sqrt{x^2 + y^2} \) and \( x = 1 + y \).
\[
\begin{align*}
z &= \sqrt{(1+y)^2 + y^2} \\
x &= 1 + y \\
y &= y
\end{align*}
\]
\[
\begin{align*}
\chi &= 1 + t \\
y &= t \\
z &= \sqrt{(1+t)^2 + t^2}.
\end{align*}
\]