Math 283: Quiz 11 (Take-home; due in class on Tues. Dec 5)

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Please use Maple, textbook, or notes if you like. You do not need to carry out the integrals here, unless otherwise specified. Note that there are 3 problems; 2 more on the back.

1. (Section 13.4) We define \( \mathbf{F}(x,y) = (x^3 + 4xy)i + (4xy - y^3)j \). Answer the following questions regarding \( \mathbf{F} \).

(a) Is \( \mathbf{F} \) conservative?

\[
\frac{\partial P}{\partial y} = 4 \quad \frac{\partial Q}{\partial x} = 4 \quad \text{not equal, so } \mathbf{F} \text{ is NOT conservative. (So don't look for } \Phi \text{ for } \mathbf{F}).
\]

(b) Set up the line integral of \( \mathbf{F} \) along the triangle whose vertices are \((0,0), (1,0), \) and \((0,1), \) traced counterclockwise, using the definition.

Along \( C_1 : \frac{\partial}{\partial x} F_1 = 0 \) \( 0 \leq t \leq 1 \). 
\[
\int_{C_1} F \cdot d\mathbf{r} = \int_0^1 (t^3 + 4t \cdot 0, 4t \cdot 0 - 0^3) \cdot <1,0> \, dt = \int_0^1 t^3 \, dt.
\]

Along \( C_2 : \frac{\partial}{\partial y} F_2 = 0 \) \( 0 \leq t \leq 1 \). 
\[
\int_{C_2} F \cdot d\mathbf{r} = \int_0^1 (1-t)^3 + 4(1-t) \cdot t, 4(1-t)t-t^3\cdot 1 \, dt = \int_0^1 (-1 + 3t - 3t^2) \, dt.
\]

Along \( C_3 : \frac{\partial}{\partial x} F_3 = 0 \) \( 0 \leq t \leq 1 \). 
\[
\int_{C_3} F \cdot d\mathbf{r} = \int_0^1 (0^3, 4 \cdot 0 - 0^3) \cdot <0,1> \, dt = \int_0^1 (-t^3) \, dt.
\]

So 
\[
\int_c F \cdot d\mathbf{r} = \int_0^1 (t^3, -1 + 3t - 3t^2 + 7t^3) \, dt = \int_0^1 (-t^3 - 3t^2 + 1 + 2t^3) \, dt = 0.
\]

(c) Find the line integral of \( \mathbf{F}(x,y) = (x^3 + 4xy)i + (4xy - y^3)j \) along the triangle whose vertices are \((0,0), (1,0), \) and \((0,1), \) traced counterclockwise, using Green's theorem.

\[
\int_c F \cdot d\mathbf{r} = \int_0^1 \int_0^{1-x} (4y - 4x) \, dy \, dx = \int_0^1 (2y^2 - 4xy) \bigg|_0^{1-x} \, dx
\]

\[
= \int_0^1 \left[ 2(1-x)^2 - 4x(1-x) \right] \, dx = \int_0^1 (2 + 6x^2 - 8x) \, dx = 0.
\]
2. (Section 13.4) Another application of Green’s Theorem is to find the area enclosed by a closed curve. The area \(A\) of region \(D\) enclosed by a simple closed curve \(C\) is

\[
A = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2} \int_C x\,dy - y\,dx \quad \text{(this is equation 5 on p. 936)}.
\]

Use one of the above formulas to find the area under one arch of the cycloid \(x = t - \sin t,\ y = 1 - \cos t\), shown below. What is the parameter range to capture one arch?

Since \(C_1\) is cycloid traced backwards, \(t\) goes from \(2\pi\) to 0.
Along \(C_1\):
\[
x' = 1 - \cos t, \quad y' = t - \sin t.
\]
So \[
\int_{C_1} x\,dy = \int_0^{2\pi} (t\sin t)\,dt
\]
Along \(C_2\):
\[
x = t, \quad y = 1 - \cos t, \quad 0 \leq t \leq 2\pi.\]
So \[
\int_{C_2} x\,dy = \int_0^{2\pi} t\,dt = \frac{2\pi^2}{2}.
\]

The area \(A\) is

\[
A = \int_0^{2\pi} \sin t \, dt - \int_0^{2\pi} t\sin t\,dt = \left[ -\cos t \right]_0^{2\pi} + \left[ t\cos t \right]_0^{2\pi} = \frac{1}{2} \cdot 2\pi + 2\pi = 3\pi.
\]

3. (Section 13.5) We look at the curl and divergence of the vector field \(G(x, y, z) = yi - 2j + xk\), whose plot is figure 11 on p. 908 in your textbook.

(a) What is the curl of \(G\)? Does your answer make sense looking at the plot?

\[
\text{curl } \mathbf{G} = \begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
y & -2 & x
\end{vmatrix} = -\mathbf{j} - \mathbf{k}.
\]

(b) What is the divergence of \(G\)? Does your answer make sense looking at the plot?

\[
\text{div } \mathbf{G} = \frac{\partial}{\partial x} y + \frac{\partial}{\partial y} (-2) + \frac{\partial}{\partial z} (x) = 0.
\]