Math 283: Quiz 7 (Take-home; due in class on Oct. 31)

Some practice finding local min, max, and saddles...

1. We have \( f(x, y) = x^2 + y^2 + x^2y - 4 \). Find the local maximum and minimum values and saddle point(s) of \( f(x, y) \), following the steps.

   (a) Find all the first order partial derivatives. You should have as many as you have the number of independent variables, so two of them in this case.

   \[
   f_x(x, y) = 2x + 2xy, \quad f_y(x, y) = 2y + x^2
   \]

   In other words, \( \nabla f(x, y) = \langle 2x + 2xy, 2y + x^2 \rangle \)

   (b) Solve \( \nabla f(x, y) = 0 \). That is, set \( f_x(x, y) = 0 \) and \( f_y(x, y) = 0 \) and solve for \( x \) and \( y \).

   \[
   2x + 2xy = 0 \Rightarrow x(1 + y) = 0 \quad \text{so} \quad x = 0 \text{ or } y = -1
   
   2y + x^2 = 0 \Rightarrow y = -\frac{x^2}{2}
   
   \text{If } x = 0 \text{ then } y = 0
   
   \text{If } y = -1 \text{ then } x^2 = 2 \quad \text{so } \quad x = \pm \sqrt{2}
   \]

   (c) The critical points are of the form \((a, b)\), where \( x = a \) and \( y = b \) are numbers you obtained in part (b). List your critical points.

   \[
   (0, 0) \quad \text{and} \quad \left( \sqrt{2}, -1 \right), \left( -\sqrt{2}, -1 \right)
   \]

   (d) Find \( f_{xx}(x, y) \), \( f_{yy}(x, y) \), and \( f_{xy}(x, y) \).

   \[
   f_{xx}(x, y) = 2 + 2y \quad f_{yy}(x, y) = 2 \quad f_{xy}(x, y) = 2x
   \]

   (e) Complete the following table by evaluating each function at each critical point.

   \[
   \begin{array}{c|c|c|c|c|c}
   \text{Critical pts.} & f & f_{xx} & f_{yy} & f_{xy} & D \quad \text{Classification} \\
   \hline
   (0, 0) & -4 & 2 & 2 & 0 & 4 & D > 0 \text{ & } f_{xx} > 0 \text{ so min } \\
   (\sqrt{2}, -1) & -3 & 0 & 2 & 2\sqrt{2} & -8 & D < 0 \text{ saddle } \\
   (-\sqrt{2}, -1) & -3 & 0 & 2 & -2\sqrt{2} & -8 & D < 0 \text{ saddle }
   \end{array}
   \]
2. We find absolute minimum and maximum of 
\[ f(x, y) = x^2 + y^2 + x^2y - 4 \] in problem 1
on the triangle whose vertices are \((-2, 2), (2, -2), \) and \((-2, -2).\)

(a) Are the critical points in problem 1 included in our set?

(b) Analyze the behavior of \( f(x, y) \) along the boundaries \( B_1, B_2, \) and \( B_3 \) shown above.

i. Along \( B_1: f(x, y) = y^2 + 4y, \) so the maximum value is \( 12 \) at \((-2, 2).\)

ii. Along \( B_2: f(x, y) = x^2 - 2x^2, \) so the maximum value is \( 0 \) at \((0, 2).\)

iii. Along \( B_3: f(x, y) = -x^2 + 2y^2, \) so the maximum value is \( 12 \) at \((-2, 2).\)

(c) The absolute maximum is the largest value listed above, and it is \( 12 \) at \((-2, 2).\)

The absolute minimum is the smallest value listed above, and it is \(-4\) at \((0, 0)\).

3. We find the maximum and minimum values of \( f(x, y, z) = x^2y^2z^2 \) subject to the constraint 
\[ x^2 + y^2 + z^2 = 1. \]

(a) Find the gradient of \( f. \) \( \nabla f(x, y, z) = \langle 2x^2y^2z^2, 2x^2yz^2, 2x^2yz^2 \rangle \)

(b) Find the gradient vector of the constraint function \( g(x, y, z) = x^2 + y^2 + z^2. \)

\[ \nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle \]

(c) Solve the set of equations \( f_x(x, y, z) = \lambda g_x(x, y, z), f_y(x, y, z) = \lambda g_y(x, y, z), f_z(x, y, z) = \lambda g_z(x, y, z), \) and \( x^2 + y^2 + z^2 = 1 \) for \( x, y, \) and \( z. \) These points may be where \( f \) attains extreme values.

One way: multiply \( 1 \) by \( x, 2 \) by \( y, 3 \) by \( z \) then

\[ 2x^2y^2z^2 = \lambda x^2, \]
\[ 2x^2y^2z^2 = \lambda y^2, \]
\[ 2x^2y^2z^2 = \lambda z^2. \]

Plug into \( 3! \) \( 3x^2 = 1 \) so \( x = \pm \frac{1}{\sqrt{3}}, y = \pm \frac{1}{\sqrt{3}}. \)

(d) Evaluate \( f \) at each of the points in part (c). The maximum value of \( f \) is \( \frac{1}{27} \) attained at \( \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right). \) The minimum value of \( f \) is \( 0 \) attained at \( \left( 0, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right), \) \( \left( \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}} \right), \) and \( \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0 \right). \)