Fall 2003 Math 283 Section 6 Exam 3 A

Name: 

If you cannot complete a problem (perhaps because you forgot a formula) but you think you know how, please describe. Correct methods will receive partial credits.

1. Determine whether each statement is true or false and circle your answer (2 points for correct answer, 1 for "I don't know", and 0 for wrong answer).

(a) The minimum value of the function \( f(x, y) = x^2 + y^2 \) subject to the constraint \( g(x, y) = x^4 + y^4 - 1 = 0 \).

\[ \text{True} \quad \text{False} \quad \text{I don't know} \]

The contour diagram shows that the minimum value is 0.

(b) If \( f(x, y) \) has two local maxima, then \( f \) must have a local minimum.

\[ \text{True} \quad \text{False} \quad \text{I don't know} \]

The contours of \( f(x, y) = x^3 - 3xy + 3y^3 \) are shown on the right.

(c) Using the figure as an aid, find all critical points of \( f \).

\[ \begin{align*}
f_x &= 0 \Rightarrow 3x^2 - 3y = 0 \Rightarrow x^2 = y \\
f_y &= 0 \Rightarrow 9y^2 - 3x = 0 \Rightarrow x = 3y^2 \\
\end{align*} \]

So \( x = y \). \( y(9y^3 - 1) = 0 \Rightarrow y = 0 \) or \( y = \frac{1}{9} \).

\[ x = 0 \quad \text{or} \quad x = 3 \left( \frac{1}{9} \right)^{\frac{1}{3}} C_1(0, 0) \quad \text{and} \quad C_2 \left( \frac{1}{9} \right)^{\frac{1}{3}} (\frac{1}{9})^{\frac{1}{3}} \]

(b) Using the second derivative test, classify the critical points found in part (a).

\[ \begin{align*}
f_{xx} &= 6x \\
f_{yy} &= 18y \\
f_{xy} &= -3 \\
D(0, 0) &= (3)^2 = 9 \quad (0, 0) \text{ is a saddle.} \\
D \text{ at the other CP is positive, and } f_{xx} > 0 \text{ so } (3 \left( \frac{1}{9} \right)^{\frac{1}{3}}, \left( \frac{1}{9} \right)^{\frac{1}{3}}) \text{ is a min.} \\
\end{align*} \]

(c) Doing any further work, at what point(s) in \([-1, 2] \times [-1, 2]\) does \( f \) attain the absolute maximum value? the absolute minimum value?

Looking at the contours, abs. max is at \((-1, 2)\)

abs. min is at \( \left( 3 \left( \frac{1}{9} \right)^{\frac{1}{3}}, \left( \frac{1}{9} \right)^{\frac{1}{3}} \right) \).
6. Set up the integral to find the surface area of parametrized surface \( \mathbf{r}(u, v) = \langle uv, u+v, u-v \rangle \), where \( u^2 + v^2 \leq 1 \), following the steps.

(a) Find the two partial derivatives \( \mathbf{r}_u \) and \( \mathbf{r}_v \).

\[
\mathbf{r}_u = \langle v, 1, 1 \rangle \quad \mathbf{r}_v = \langle u, 1, -1 \rangle
\]

(b) Find \( \mathbf{r}_u \times \mathbf{r}_v \) and \( |\mathbf{r}_u \times \mathbf{r}_v| \).

\[
\begin{align*}
\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v & 1 & 1 \\ u & 1 & -1 \end{vmatrix} \\
&= (-2)\mathbf{j} - (-u-u)\mathbf{j} + (u-u)\mathbf{k} \\
&= -2\mathbf{j} + (v+u)\mathbf{j} + (u-u)\mathbf{k} \\
|\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{4 + (v+u)^2 + (v-u)^2} \\
&= \sqrt{4 + 2v^2 + 2u^2} = \sqrt{4 + 2(v^2 + u^2)}
\end{align*}
\]

(c) Set up the integral in coordinate system of your choice. Do not evaluate.

\[
\int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} \sqrt{4 + 2(v^2 + u^2)} \, du \, dv
\]

In polar coordinates,

\[
\int_{0}^{2\pi} \int_{0}^{1} \sqrt{4 + 2r^2} \, r \, dr \, d\theta
\]
3. Evaluate the following integral.

\[
\int_0^1 \int_0^x (x + 2y) dy \, dx = \int_0^1 \left( xy + y^2 \right) \bigg|_0^1 \, dx = \int_0^1 (1 + 1) \, dx = 2.
\]

4. Set up the double integral that represents the volume of the solid bounded by the paraboloid \( z = x^2 + y^2 + 4 \) and the planes \( x = 0, y = 0, z = 0 \), and \( x + y = 1 \). Do not evaluate.

\[
\text{Volume} = \iiint_D \text{height} \, dV = \int_0^1 \int_0^{y+1-x} (x^2 + y^2 + 4) \, dy \, dx
\]

5. Rewrite the integral \( \iint_D \sqrt{x^2 + y^2} \) where \( D \) is the region bounded by the semicircle \( x = \sqrt{4 - y^2} \) and the y-axis, in polar coordinates. Do not evaluate.

\[
\int_{-\pi/2}^{\pi/2} \int_0^{\sqrt{4 - y^2}} r \, dr \, d\theta
\]