1. **Continuous annuities.** If the payments are being made continuously at the rate \( f(t) \) at exact moment \( t \), then the present value of an \( n \)-period continuous varying annuity is

\[
\int_0^n f(t)e^{-\int_0^t \delta_r \, dr} \, dt,
\]

where \( \delta_r \) is the force of interest. Under compound interest this becomes

\[
\int_0^n f(t)v^t \, dt.
\]

Special cases:

- If \( f(t) = 1 \) then the present value is

\[
\overline{a}_n = \int_0^n v^t \, dt = \frac{1-v^n}{\delta}.
\]

The accumulated value is

\[
\overline{s}_n = (1+i)^n \overline{a}_n = \int_0^n (1+i)^{n-t} \, dt = \frac{(1+i)^n - 1}{\delta}.
\]

- If \( f(t) = t \) (continuously increasing annuity) then the present value is

\[
\overline{I}_n = \int_0^n tv^t \, dt = \overline{a}_n - nv^n.
\]

2. **Varying annuity - arithmetic progression.** Annuity immediate with a term of \( n \) periods in which payments begin at \( P \) and increase by \( Q \) per period thereafter:

Present Value = \( Pa_{\overline{n}|} + Q\overline{a}_{\overline{n}|} - n\overline{v}_n \).

Accumulated Value = \( Ps_{\overline{n}|} + Q\overline{s}_{\overline{n}|} - n \).

Special case \( P = Q = 1 \):

Present Value = \( \overline{I}a_{\overline{n}|} = \overline{a}_{\overline{n}|} - n\overline{v}_n \).

Accumulated Value = \( \overline{I}s_{\overline{n}|} = \overline{a}_{\overline{n}|}(1+i)^n = \overline{s}_{\overline{n}|} - n \).

Special case \( P = n, Q = -1 \):

Present Value = \( \overline{D}a_{\overline{n}|} = \frac{n-a_{\overline{n}|}}{i} \).

Accumulated Value = \( \overline{D}s_{\overline{n}|} = \overline{D}a_{\overline{n}|}(1+i)^n = \frac{n(1+i)^n - \overline{s}_{\overline{n}|}}{i} \).

3. **Varying annuity - geometric progression.** The present value of an annuity immediate with a term of \( n \) periods in which the first payment is 1 and successive payments increase in geometric progression with a common ratio \( 1+k \) is

\[
\frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k}, \quad i \neq k.
\]
4. **Dollar-weighted rate of interest.** The rate \( i \) is the solution of an exact equation of value:

\[
I = iA + \sum_t C_t \cdot 1-t \cdot i_t,
\]

where

\[
I = B - A - C - \text{the interest.}
\]

\( A \) - The amount in the fund at the beginning of the period.

\( B \) - The amount in the fund at the end of the period.

\( C_t \) - The net amount of principal contributed at time \( t \).

\( C \) - The total amount of principal contributed during the period.

\( 1-t \cdot i_t \) - the rate of interest for the period \((t, 1)\). Under compound interest use \( 1-t \cdot i_t = (1 + i)^{1-t} - 1 \). For an approximation (simple interest) use \( 1-t \cdot i_t = (1 - t)i \).

5. **Time weighted rate of interest.** Here, the rate \( i \) is equal to

\[
i = (1 + j_1)(1 + j_2)\cdots(1 + j_m) - 1,
\]

where

\[
j_k = \frac{B'_k}{B'_{k-1} + C'_{k-1}}, \quad k = 1, 2, \ldots, m,
\]

is the rate over the subinterval \((t_{k-1}, t_k)\) and

\( C'_k \) - The net amount of principal contributed at time \( t_k \).

\( B'_k \) - The fund value (immediately before each contribution) at time \( t_k \).

6. **Loans with varying series of payments.** Basic equation of value:

\[
L = \sum_t v^t R_t,
\]

where

\( L \) - The loan amount.

\( R_t \) - installment payment (including principal and interest) at time \( t \).

For the sinking fund method we have:

\[
L = \sum_t R_t (1 + j)^{n-t} - iL_{s, j},
\]

where \( L \) and \( R_t \) are as above and

\( i \) - rate of interest paid on the loan.

\( j \) - rate of interest earned on the sinking fund.