1. Suppose that $X_1, \ldots, X_n$ form a random sample from a gamma distribution with density function
\[
f(x|\alpha, \beta) = \frac{x^{\alpha-1}e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} \quad \text{for } x > 0 \text{ and zero otherwise},
\]
where the value of $\alpha > 0$ is known while the value of $\beta > 0$ is unknown. Show that the joint density of $X_1, \ldots, X_n$ has a monotone likelihood ratio in the statistic $\bar{X} = \frac{1}{n}\sum X_i$.

Thus, you need to show that for any $0 < \beta_1 < \beta_2$ the ratio
\[
\frac{f_n(x_1, \ldots, x_n|\alpha, \beta_2)}{f_n(x_1, \ldots, x_n|\alpha, \beta_1)}
\]
is an increasing function of the above statistic.

2. In the setting of Problem 1 with $\alpha = 1$, show that the best test for
\[
H_0 : \beta = \beta_0 \quad H_1 : \beta = \beta_1
\]
given by Neyman-Pearson lemma has the same critical region for every $\beta_1 > \beta_0$. Using the above, argue that there exists a UMP test for the problem
\[
H_0 : \beta = \beta_0 \quad H_1 : \beta > \beta_0,
\]
and it rejects $H_0$ if $\sum X_i \geq c$ for some constant $c$. Assuming that $\alpha = 1$, $\beta_0 = 2$, $n = 10$, and $\alpha = 0.01$ (the level of significance), determine the constant $c$. [Hint: Relate $\sum X_i$ to chi-square distribution.]

3. Suppose that $X_1, \ldots, X_n$ form a random sample from a normal distribution with an unknown mean $\theta$ and a given variance $\sigma^2$. Suppose that the following hypotheses are to be tested:
\[
H_0 : \theta \geq \theta_0 \quad H_1 : \theta < \theta_0,
\]
where $\theta_0$ is a specified constant. Show that the likelihood ratio test rejects $H_0$ if $\bar{X} \leq k$, where $k$ is some constant.
4. Suppose that the significance level of the test given in the previous problem is to be $\alpha$. Show if the constant $k$ is chosen so that

$$Pr(\bar{X} \leq k|\theta = \theta_0) = \alpha,$$

then the size of this test is equal to $\alpha$ (that is show that $Pr(\bar{X} \leq k|\theta) \leq \alpha$ for any $\theta \geq \theta_0$).

5. In the setting of the previous two problems, what decision is reached if $\theta_0 = 1$, $\sigma^2 = 4$, $n = 16$, $\bar{X} = 0.5$, and $\alpha = 0.05$?