Due Thursday, February 5

Consider the function
\[
f(x) = \frac{1}{2\pi |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}, \quad x = (x_1, x_2) \in \mathbb{R}^2,
\]
where \(\mu = (\mu_1, \mu_2)'\), \(-\infty < \mu_1, \mu_2 < \infty\),
\[
\Sigma = \begin{bmatrix}
\sigma_1^2 & \rho \sigma_1 \sigma_2 \\
\rho \sigma_1 \sigma_2 & \sigma_2^2
\end{bmatrix},
\]
\(\sigma_1, \sigma_2 > 0, -1 < \rho < 1\), and \(|\Sigma|\) denotes the determinant of the matrix \(\Sigma\).

1. Show that the above function can be written as
\[
f(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)} \left\{ \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2 - 2\rho \left( \frac{x_1 - \mu_1}{\sigma_1} \right) \left( \frac{x_2 - \mu_2}{\sigma_2} \right) \left( \frac{x_2 - \mu_2}{\sigma_2} \right)^2 \right\} }.
\]

2. Show that this function is nonnegative and integrates to 1 on \(\mathbb{R}^2\). [Therefore, it is a valid probability density function for a two dimensional random vector \(X_1, X_2\). The corresponding probability distribution is called bivariate normal distribution].

3. Show that the marginal distribution of \(X_i\) is normal with mean \(\mu_i\) and variance \(\sigma_i^2\), \(i = 1, 2\).

4. Show that the covariance of \(X_1\) and \(X_2\) is \(\text{Cov}(X_1, X_2) = \rho \sigma_1 \sigma_2\), and consequently the parameter \(\rho\) is the correlation of \(X_1\) and \(X_2\). Further, show that if \(\rho = 0\) then \(X_1\) and \(X_2\) are independent.

5. Show that the marginal distribution of \(X_2\) given \(X_1 = x_1\) is normal with mean
\[
\mu_{2|1} = E(X_2 | X_1 = x_1) = \mu_2 + \rho \sigma_2 \left( \frac{x_1 - \mu_1}{\sigma_1} \right)
\]
and variance

$$\sigma^2_{2|1} = Var(X_2|X_1 = x_1) = (1 - \rho^2)\sigma^2_2.$$ 

By symmetry, argue that similar property is shared by the other conditional distribution: the conditional distribution of $X_1$ given $X_2 = x_2$ is normal with mean

$$\mu_{1|2} = E(X_1|X_2 = x_2) = \mu_1 + \rho \sigma_1 \left( \frac{x_2 - \mu_2}{\sigma_2} \right)$$

and variance

$$\sigma^2_{1|2} = Var(X_1|X_2 = x_2) = (1 - \rho^2)\sigma^2_1.$$