Due Wednesday, November 16

1. Let $N(t)$ be a nonhomogeneous Poisson process with intensity function $\lambda(t)$ and the mean value function

$$m(t) = \int_0^t \lambda(s)ds.$$ 

Using the relation

$$Pr(X_\lambda \geq n) = Pr(G_n \leq \lambda),$$

where $X_\lambda$ is a Poisson random variable with mean $\lambda$ and $G_n$ is a gamma variable with p.d.f.

$$f_n(x) = \frac{x^{n-1}e^{-x}}{\Gamma(n)}, \quad x > 0,$$

show that the p.d.f. of $S_n$ - the $n$th arrival time of the process $N(t)$ is

$$f_n(t) = \frac{\lambda(t)}{\Gamma(n)} [m(t)]^{n-1}e^{-m(t)}.$$

What is the distribution of the waiting time for the first arrival if $\lambda(t) = (1 + t)^{-1}$?

2. Let $N(t)$ be a Poisson process with intensity $\lambda$, and let $Y_1, Y_2, \ldots$ be i.i.d. random variables with mean $\mu$ and variance $\sigma^2$. Derive the mean and the variance of the compound Poisson random variable

$$X(t) = \sum_{i=1}^{N(t)} Y_i.$$ 

3. Events occur according to a nonhomogeneous Poisson process whose mean value function is given by

$$m(t) = t^2 + 2t, \quad t \geq 0.$$ 

What is the probability that $n$ events occur between times $t = 4$ and $t = 5$?
4. Chapter 5, No. 85 (Same in 9th and 10th editions)

5. Let $N_1, \ldots, N_n$ be independent Poisson variables with means $\lambda_1, \ldots, \lambda_n$, and let for each $i$ ($1 \leq i \leq n$) the random variables $Y_j^{(i)}$, $j = 1, 2, \ldots$ be i.i.d. with moment generating functions $M_i(t)$. Thus, the quantities

$$X_i = \sum_{j=1}^{N_i} Y_j^{(i)}, \quad i = 1, \ldots, n$$

are independent compound Poisson variables. Show that $X = X_1 + \cdots + X_n$ has a compound Poisson distribution as well, and admits the representation

$$X = \sum_{j=1}^{N} Y_j,$$

where $N$ is a Poisson variable with mean $\lambda = \lambda_1 + \cdots + \lambda_n$ and the $Y_j$'s are i.i.d. with the m.g.f.

$$M(t) = \sum_{j=1}^{n} p_j M_j(t), \quad p_j = \frac{\lambda_j}{\lambda}.$$
\[ F_n(t) = P(S_n \leq t) = P(N(t) \geq n) = \int_0^t \lambda(t) dt = m(t) \]

So the density of \( S_n \) is

\[ \frac{d}{dt} F_n(t) = \frac{d}{dt} P(G_n \leq m(t)) = f_n(m(t)) m'(t) \]

\[ = \frac{[m(t)]^{n-1} e^{-m(t)}}{m(n)} \cdot \lambda(t) \]

\[ m'(t) = \frac{d}{dt} \int_0^t \lambda(x) dx = \lambda(t) \]

If \( \lambda(t) = (1+t)^{-1} \) then \( m(t) = \int_0^t (1+x)^{-1} dx = \ln(1+t) \]

With \( n=1 \) the pdf is

\[ \frac{[\ln(1+t)]^{n-1} e^{-\ln(1+t)}}{m(n)} \cdot (1+t)^{-1} = \frac{1}{(1+t)^2} \]
2. \( N(t) \) - Poisson process

\[ Y_i \sim \text{iid} \quad EY_i = \mu \quad \text{Var} Y_i = \sigma^2. \]

\[ EN(t) = \lambda t, \quad \text{Var} N(t) = \lambda t. \]

\[ E[X(t)] = \sum_{i=0}^{N(t)} Y_i \]

\[ \text{Var} X(t) = EN(t) \text{Var} Y_i + (EY_i)^2 \text{Var} N(t) \]

\[ = \lambda t \sigma^2 + \mu^2 \lambda t = \lambda t \left( \sigma^2 + \frac{\mu^2}{\lambda} \right) \]

3. Chapter 5 #77 (9th Ed)

\( N(5) - N(4) \sim \text{Poisson with mean } \lambda = \int_4^5 \lambda(t) \, dt \)

Now \( m(t) = \int_0^t \lambda(y) \, dy = t^2 + 2t \)

\[ m'(t) = 2t + 2 = \frac{\lambda(t)}{t} \]

\[ \lambda = \int_4^5 \lambda(t) \, dt = \int_4^5 2t + 2 \, dt = [t^2 + 2t]_4^5 = 5^2 + 10 - 4^2 - 8 = 25 + 10 - 16 - 8 = 11 \]

\[ \Pr (N(5) - N(4) = n) = \frac{e^{-11} \cdot 11^n}{n!} \]
Chapter 5 # 85

\[ N(t) \to \text{Poisson process with } \lambda = 5 \text{ (per week)} \]

\[ X_i = \exp \text{ (mean = 2000)} \]

\[ Y = \sum_{i=1}^N N_i(t) \]

\[ EY = EN(4) EX_i = 4.5 \times 2000 = 90,000 \]

\[ \text{Var Y} = EN(4) \text{Var } X_i + \frac{\text{Var } X_i^2}{N(4)} \text{Var } N(4) \]

\[ = 4.5 \times 2000^2 + 2000^2 \times 4.5 \]

\[ = 2 \times 20,200^2 = 16,000,000 = 1.6 \times 10^8 \]

\[ M = \text{mf of } X_1 + \cdots + X_n \]

\[ M(t) = \mathbf{E} e^\sum_{i=1}^N y_i(t) \]

where \( y_i(t) \) is \( \text{mf of } X_i \)

\[ \bar{y}_i(t) = \mathbf{E} e^\sum_{j=1}^{N_i} y_j(t) = M_{N_i}(\log \bar{y}_j(t)) = M_{N_i}(\log M_i(t)) \]

For Poisson distribution with mean \( \lambda_i \), the \( \text{mf is} \)

\[ M_{N_i}(t) = e^{\lambda_i(e^t - 1)} \]

\[ M_{N_i}(\log M_i(t)) = e^{\lambda_i(H_i(t) - 1)} \]

\[ M(t) = \frac{1}{n} \sum_{i=1}^n \bar{y}_i(t) = \frac{1}{n} e^{\lambda_i(H_i(t)-1)} = e^{E(\lambda_i(H_i(t)-1))} \]
Now we do the mgf of \( \sum_{j=1}^{N} Y_j \)

This mgf is

\[
HN (\log Y_j \alpha (t)) =
\]

\[
= HN (\log \sum_{j=1}^{n} p_j Y_j \alpha (t))
= e^{\lambda (e^{\lambda \sum_{j=1}^{n} \frac{X_j}{\lambda} Y_j \alpha (t)} - 1)}
= e^{\lambda \left( \sum_{j=1}^{n} p_j Y_j \alpha (t) \right) - \lambda \left( \sum_{j=1}^{n} \frac{X_j}{\lambda} Y_j \alpha (t) - 1 \right)}
= e^{\sum_{j=1}^{n} \lambda p_j Y_j \alpha (t) - \sum_{j=1}^{n} \lambda \frac{X_j}{\lambda} Y_j \alpha (t)}
= e^{\sum_{j=1}^{n} \lambda p_j Y_j \alpha (t) - \sum_{j=1}^{n} \lambda X_j}
= e^{\sum_{j=1}^{n} \lambda (Y_j \alpha (t) - 1)}
\]

which coincides with the mgf obtained previously for

\( X_1 + X_2 + \ldots + X_N \)