1. Suppose $X_1, \ldots, X_n$ are i.i.d. random variables having a normal distribution with known mean $\mu$ and unknown variance $\theta$. Let the prior distribution of $\theta$ have the following pdf:

$$\lambda(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\beta/\theta}, \quad \theta > 0,$$

where $\alpha > 0$ and $\beta > 0$ are the parameters. Calculate the density of the posterior distribution of $\theta$. [Hint: For any $\alpha > 0$ and $\beta > 0$ the above function is a probability density function - so it integrates to 1 on $(0, \infty)$].

2. In the setting of Problem 1, show that if the squared error loss function is used and the sample size $n$ is larger than 2, then the Bayes estimator of $\theta$ is

$$\delta = \delta(X_1, \ldots, X_n) = \frac{\beta \frac{1}{2} \sum (X_i - \mu)^2}{\alpha + n/2 - 1}.$$

3. Calculate the risk of the estimator given in Problem 2 (assuming the squared error loss function). [Hint: The random variable $S^2 = \sum (X_i - \mu)^2/\theta$ has a chi-square distribution with $n$ degrees of freedom, so the mean and variance of $S^2$ are $n$ and $2n$.]

4. Suppose that the proportion $\theta$ of defective items in a large lot is unknown, and that the prior distribution of $\theta$ is given by the following density:

$$\lambda(\theta) = 2(1 - \theta), \quad 0 < \theta < 1.$$

In a random sample of eight items exactly three are found to be defective. Determine the posterior distribution of $\theta$. What is the Bayes estimator of $\theta$?

5. Suppose that $X_1, \ldots, X_n$ are i.i.d. Bernoulli random variables with parameter $\theta$ [that is $Pr(X_i = 1) = \theta$ and $Pr(X_i = 0) = 1 - \theta$]. Show that if the loss function is

$$L(\theta, \delta) = \frac{(\theta - \delta)^2}{\theta(1-\theta)},$$
then $\delta^* = \frac{1}{n} \sum X_i$ (the sample mean) is the minimax estimator of $\theta$.

[Hint: You may want to follow these steps:

(i) Take the prior distribution of $\theta$ to be standard uniform and derive the posterior distribution of $\theta$, $\lambda(\theta|x_1,\ldots,x_n)$.

(ii) Find the Bayes estimator of $\theta$. Note that it will not be given by the mean of the posterior distribution, since the loss function is not the square error loss. You should directly minimize the expected value $\int L(\theta, \delta) \lambda(\theta|x_1,\ldots,x_n)d\theta$. Think of this integral as $E(\theta - \delta)^2$ where the expected value is taken with respect to a certain beta distribution.

(iii) Show that your Bayes estimator has constant risk].