1. Suppose $X_1, \ldots, X_n$ are i.i.d. random variables from a normal distribution with known mean $\mu$ and an unknown variance $\sigma^2 > 0$, and suppose that it is desired to test the following hypotheses:

$$H_0 : \sigma^2 \leq \sigma_0^2 \ vs. \ H_1 : \sigma^2 > \sigma_0^2,$$

where $\sigma_0^2$ is a given positive constant.

(a) Show that the joint density of the $X_i$’s has a monotone likelihood ratio in the statistic $T = \sum_{i=1}^{n} (X_i - \mu)^2$.

(b) Argue that at every level of significance $\alpha$ there exists a UMP test of these hypotheses, and it rejects $H_0$ whenever $T \geq C$ for some constant $C$.

(c) Show that if the constant $C$ is chosen such that

$$Pr(T \geq C | \sigma^2 = \sigma_0^2) = \alpha,$$

then the above test is a level $\alpha$ test.

(d) Determine the value of $C$ using the above condition when $\alpha = 0.05$, $n = 10$, $\sigma_0^2 = 2$, and $\mu = 0$.

2. Section 8.6, No. 2
3. A single observation $X$ from a distribution with density function

$$f(x|\theta) = \begin{cases} \theta x^{\theta-1} & X > 1, \\ 0 & \text{otherwise} \end{cases}$$

is used to test the null hypothesis $H_0 : \theta = 2$ against the alternative $H_1 : \theta = 4$.

(a) What is the parameter space $\Omega$? What is the subset $\Omega_0$ corresponding to the null hypothesis?

(b) Show that the critical region of the likelihood ratio test based on the statistic

$$\Lambda = \frac{\sup_{\theta \in \Omega_0} f(X|\theta)}{\sup_{\theta \in \Omega} f(X|\theta)}$$

rejects $H_0$ if $X \leq K$ for some constant $K$.

(c) Show that the critical region of the best (Neyman-Pearson) test is the same.

(d) If the probability of Type I error of the above test is $3/4$, then what is the probability of Type II error?

4. Let $X_1, \ldots, X_n$ be a random sample from exponential distribution with mean $1/\beta$, $\beta > 0$. Consider the problem of testing $H_0 : \beta \leq \beta_0$ vs. $H_1 : \beta > \beta_0$, where $\beta_0 > 0$ is a given constant.

(a) Show that the likelihood ratio test has the critical region of the form: $\bar{X} \leq K$ for some constant $K$.

(b) Show that for any given $0 < K < 1/\beta_0$, the power function $\pi(\beta)$ of the above test is increasing in $\beta$.

(c) Use the above to show that Type I error is maximized when $\beta = \beta_0$. [So that when the constant $K$ is chosen so that $\pi(\beta_0) = \alpha$, then this will be a level $\alpha$ test.]

(d) Use the above to construct a level $\alpha = 0.05$ test (that means find the critical value $K$) when $\beta = 1/2$ and $n = 10$. [Hint: Relate $\bar{X}$ to chi-square distribution].

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5. Suppose that a single observation $X$ is taken from a distribution with density function

$$f(x|\theta) = \begin{cases} 2(1-\theta)x + \theta & 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \leq \theta \leq 2$ is an unknown parameter.

(a) Suppose that the following hypotheses are to be tested

$$H_0 : \theta = \theta_0 \ vs. \ H_1 : \theta = \theta_1,$$

where $0 \leq \theta_1 < \theta_0 \leq 2$. Show that for any $0 < \alpha < 1$ there exist a test $\delta$ of size $\alpha$ that minimizes the probability of Type II error (among all level $\alpha$ tests), and its critical region is of the form: $X > K$ for some constant $K$. Further, show that $K$ does not depend on $\theta_1$.

(b) Suppose now that it is desired to test

$$H_0 : \theta = 1 \ vs. \ H_1 : \theta < 1.$$

Explain why for any $0 < \alpha < 1$ there exist a UMP level $\alpha$ test for this problem, and it rejects the null hypothesis if $X > 1 - \alpha$.

6. In the setting of Problem 5, consider the following hypotheses to be tested:

$$H_0 : \theta \geq 1 \ vs. \ H_1 : \theta < 1. \quad (1)$$

(a) Determine the power function of the test that rejects the above null hypothesis if $X > 1 - \alpha$ (this is the same test as that derived in Problem 5b).

(b) Show that this is a level $\alpha$ test for problem (1).

(c) Is this a UMP test for problem (1)?

**BONUS PROBLEM FOR EXTRA CREDIT**

7. In the setting of Problem 1, show that the likelihood ratio test rejects $H_0$ whenever $\hat{\sigma}^2 \geq K$ for some constant $K$, where $\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \mu)^2$ is an MLE of $\sigma^2$ (so that this test is equivalent to the UMP test).