Some historical markers

-1930: John von Neumann (the same Hungarian guy from computer science, game theory and economics) defines rings of operators, now called von Neumann algebras.

-1943: Gelfand and Neumark show that Banach algebras with an adjoint $*$ satisfying certain properties (now called C*-algebras) are $*$ isomorphic to norm-closed self-adjoint subalgebras of $B(H)$.

-1947: Segal relates harmonic analysis on locally compact groups to (non-commutative) self-adjoint operator algebras acting on a Hilbert space. Also, Segal treats the foundations of quantum mechanics from the point of view of operator algebras.

-1950’s: The elaboration of the viewpoint that C*-algebras are non-commutative versions of $C_0(X)$ and von Neumann algebras are non-commutative versions of $L^\infty(X, \mu)$. 
-1967: Haag, Hugenholtz and Winnink relate quantum statistical mechanics and operator algebras by introducing the KMS condition.

-1970's: Several major ideas from algebraic topology (like K-theory and extensions) were incorporated.

-1980's: Connes starts his vast non-commutative geometry program. Jones studies subfactors and discovers a new knot invariant. Voiculescu initiates the theory of free probability and free entropy.

-1990’s and 2000’s: Big progress in the classification theory of C*-algebras in terms of K-theoretical invariants.
Definitions and Examples

A C*-algebra is a complex Banach algebra $A$ with involution $\ast$ such that $||a^*a|| = ||a||^2$. It is a non-commutative ring of operators (think infinite matrices), in fact $A \subset B(H)$ for $H$ a Hilbert space (think infinite dimensional Euclidean space).

1) $C(X) = \{ f : X \to \mathbb{C} \mid f \text{ continuous} \}$ with norm $||f|| = \sup |f(x)|$ and involution $f^*(x) = \overline{f(x)}$, where $X$ is a compact Hausdorff space. Any commutative unital C*-algebra is isomorphic to some $C(X)$.

2) $M_n$ the set of $n \times n$ matrices $M$ with complex entries, $||M|| = \sup \{|Mv| : |v| = 1\}$, where $|v|$ is the vector length, $M^* = \overline{M^t}$. Every finite dimensional C*-algebra is a direct sum of matrices.

3) $B(H)$, the bounded linear operators $T$ on a Hilbert space with involution given by $\langle Tx, y \rangle = \langle x, T^*y \rangle$.

4) Inductive limits of finite dimensional C*-algebras, like the compact operators (limits of finite rank operators)

$$\mathbb{C} \to M_2 \to M_3 \to ...$$
or the CAR algebra or Fermion algebra (used to represent the canonical anti-commutation relations for systems of particles obeying Fermi statistics)

\[ M_2 \rightarrow M_4 \rightarrow M_8 \rightarrow ..., a \mapsto \begin{bmatrix} a \\
\end{bmatrix}. \]

5) \( C(X, M_n) = \{ f : X \rightarrow M_n \mid f \text{ continuous} \} \) and inductive limits like the Bunce-Deddens algebras

\[ C(\mathbb{T}) \rightarrow C(\mathbb{T}, M_2) \rightarrow C(\mathbb{T}, M_4) \rightarrow ..., z \mapsto \begin{bmatrix} 0 & 1 \\
z & 0 \end{bmatrix}. \]

6) The convolution algebra \( C^*(G) \) of a locally compact group (or groupoid) \( G \). For \( a, b \in C_c(G) \), define

\[ (a \cdot b)(g) = \int a(h)b(h^{-1}g)dh, \]

\[ a^*(g) = \overline{a(g^{-1})}, \|a\| = \int |a(g)|dg, \]

and then take a completion.

For \( G \) abelian, \( C^*(G) \cong C_0(\hat{G}) \) by Fourier transform. For example, \( C^*(\mathbb{Z}) \cong C(\mathbb{T}) \).

For \( G = \mathbb{Z} \rtimes \mathbb{Z}_2 \) the infinite dihedral group, \( C^*(G) \) is isomorphic to

\[ C(\mathbb{T}) \rtimes \mathbb{Z}_2 \cong \{ f : [0, 1] \rightarrow M_2 \mid f(0), f(1) \text{ diagonal} \}. \]
7) Graph C*-algebras. Let \((E^0, E^1, r, s)\) be an oriented graph. Its C*-algebra is generated by mutually orthogonal projections \(P_v\) for \(v \in E^0\) and partial isometries \(T_e\) for \(e \in E^1\) such that

\[
T_e^*T_e = P_{s(e)}, \quad P_v = \sum_{r(e)=v} T_eT_e^*.
\]

**Application**

In quantum field theory, one describes a physical system by a unital C*-algebra \(A\). The self-adjoint elements of \(A\) (\(x \in A\) with \(x^* = x\)) are thought of as the observables, or the measurable quantities, of the system. A state of the system is a positive functional on \(A\) (a \(\mathbb{C}\)-linear map \(\phi : A \to \mathbb{C}\) with \(\phi(aa^*) > 0\) for all \(a \in A\)) such that \(\phi(1_A) = 1_A\). If the system is in state \(\phi\), then \(\phi(x)\) is the expected value of the observable \(x\).
References


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