6.4

1. For \( x \in [-1, 1] \) we have \( \left| \frac{x^k}{k^2} \right| \leq \frac{1}{k^2} \) and \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is convergent. By the Weierstrass \( M \)-test, the sequence of partial sums \( \{g_n\} \) converges uniformly on \([-1, 1]\) to \( f \). Since each \( g_n \) is continuous, it follows that \( f \) is continuous on \([-1, 1]\).

2. Since \( |\sin kx| \leq 1 \) and \( \sum_{k=1}^{\infty} \frac{1}{2^k} \) is convergent, the sequence of partial sums \( \{g_n\} \), which are continuous functions, converges uniformly to \( f \) on \( \mathbb{R} \). It follows that \( f \) is continuous on \( \mathbb{R} \).

4. With \( c_k = \frac{1}{k^{3k}} \) we have \( \lim_{k \to \infty} \frac{c_{k+1}}{c_k} = \lim_{k \to \infty} \frac{k^{3k}}{(k+1)^{3k+1}} = \frac{1}{3} \), hence \( R = 3 \).

6. Let \( c_k = \frac{1}{k^{\sqrt{k}}} \). Then \( \limsup_{k \to \infty} \frac{c_k^{1/k}}{c_k^{1/k}} = \lim_{k \to \infty} \frac{1}{k^{\sqrt{k}/k}} = 1 \), hence \( R = 1 \).

8. Notice that \( c_{2k} = 2^k \) and \( c_{2k+1} = 0 \). It follows that \( \limsup_{k \to \infty} c_k^{1/k} = \lim_{k \to \infty} (2^{k/2})^{1/k} = \sqrt{2} \), hence \( R = 1/\sqrt{2} \).

Or, using the ratio test, \( \lim_{k \to \infty} \frac{2^{k+1}x^{2k+2}}{2^kx^{2k}} = 2x^2 < 1 \) implies \( |x| < \frac{1}{\sqrt{2}} \), so \( R = 1/\sqrt{2} \).

10. Notice that \( |a_kx^k| \leq a_k \to 0 \) for \( x \in [0, 1] \), hence the sequence of decreasing functions \( \{a_kx^k\} \) converges uniformly to 0 on \([0, 1]\). If \( s_n(x) = \sum_{k=0}^{n} (-1)^{k+1}a_kx^k \), by Theorem 6.3.2, it follows that \( s_n(x) \) is convergent to \( s(x) \) for all \( x \in [0, 1] \). Moreover, from \( |s_n(x) - s(x)| \leq a_{n+1}x^{n+1} \) we get that \( s_n \to s \) uniformly on \([0, 1]\). Since each \( s_n \) is continuous, it follows that \( s \) is continuous on \([0, 1]\).

11. From exercise 10 with \( a_k = 1/k \), the power series \( \sum_{k=1}^{\infty} (-1)^{k-1}x^k/k \) converges uniformly to a continuous function \( s \) on \([0, 1]\). Moreover, from example 6.4.11 we have \( s(x) = \ln(1+x) \) for \( x \in [0, 1] \). Since \( s \) is continuous at 1, it follows that

\[
s(1) = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{1}{k} = \lim_{x \to 1} \ln(1+x) = \ln 2.
\]
2. Let \( f(x) = \cos x \). Since
\[
f'(x) = -\sin x, \quad f''(x) = -\cos x, \quad f^{(3)}(x) = \sin x, \quad f^{(4)}(x) = \cos x,
\]
we see that they repeat with period 4, and therefore \( f^{(k)}(0) \) takes values
\[
1, 0, -1, 0, 1, 0, -1, 0, ...
\]
It follows that
\[
\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}.
\]
Since \( |R_n(x)| = \left| \frac{\cos[(n+1)c]}{(n+1)!} x^{n+1} \right| \leq |x|^{n+1} \) \( (n+1)! \to 0 \), we see that the series \( \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \) converges to \( \cos x \) for all \( x \).

5. We have \( f(x) = (1 + x)^{1/2}, f'(x) = \frac{1}{2}(1 + x)^{-1/2}, \ f''(x) = -\frac{1}{4}(1 + x)^{-3/2} \) and in general for \( n \geq 1 \)
\[
f^{(n)}(x) = \frac{1}{2} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right) \cdots \left( \frac{1}{2} - n + 1 \right) (1 + x)^{\frac{1}{2} - n}.
\]
It follows that \( f(0) = 1 \) and for \( n \geq 1, \ f^{(n)}(0) = \frac{1}{2} \left( \frac{1}{2} - 1 \right) \left( \frac{1}{2} - 2 \right) \cdots \left( \frac{1}{2} - n + 1 \right) \) and
\[
\sqrt{1 + x} = \sum_{k=0}^{n} \left( \frac{1}{2} \right)^k x^k + R_n(x),
\]
where
\[
R_n(x) = \left( \frac{1}{2} \frac{1}{n+1} \right) (1 + c)^{\frac{1}{2} - n - 1} x^{n+1}
\]
for some \( c \) between 0 and \( x \). Note that
\[
\sum_{k=0}^{n} \left( \frac{1}{2} \right)^k x^k = 1 + \frac{1}{2} x + \sum_{k=2}^{n} \frac{(-1)^{k-1} 1 \cdot 3 \cdots (2k - 3)}{2^k \cdot k!} x^k.
\]

6. Since \( f'(x) = 3x^2 - 2x - 4, \ f''(x) = 6x - 2, \ f'''(x) = 6, \) we have \( f(1) = 0, f'(1) = -3, f''(1) = 4, f'''(1) = 6, \) so
\[
x^3 - x^2 - 4x + 4 = -3(x - 1) + \frac{4}{2!} (x - 1)^2 + \frac{6}{3!} (x - 1)^3 = -3(x - 1) + 2(x - 1)^2 + (x - 1)^3.
\]
Note that the remainder is 0.
7. Let \( f(x) = \ln(1+x) \). Then \( f'(x) = \frac{1}{1+x}, f''(x) = -\frac{1}{(1+x)^2} \) and for \( n \geq 1 \)

\[
f^{(n)}(x) = (-1)^{n-1}(n-1)! \]

We get \( f(0) = 0 \) and \( f^{(n)}(0) = (-1)^{n-1}(n-1)! \) for \( n \geq 1 \), so

\[
\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n-1}\frac{x^n}{n} + R_n(x),
\]

where

\[
R_n(x) = (-1)^{n}\frac{1}{(n+1)(1+c)^{n+1}}x^{n+1}
\]

for some \( c \) between 0 and \( x \).

8. For \(-1 < x < 1\), we have

\[
\frac{1}{\sqrt{1-x}} = (1-x)^{-1/2} = \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k (-x)^k,
\]

hence

\[
\frac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{\infty} \left( -\frac{1}{2} \right)^k (-x^2)^k.
\]

Since for \( k \geq 1 \)

\[
\left( -\frac{1}{2} \right)^k \frac{(-1/2-1)(-1/2-2)\cdots(-1/2-k+1)}{k!} = \frac{(-1)^k \cdot 1 \cdot 3 \cdots (2k-1)}{2^k \cdot k!},
\]

we obtain

\[
\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdots (2k-1)}{2^k \cdot k!} x^{2k}.
\]

By integration we get

\[
\arcsin x = x + \sum_{k=1}^{\infty} \frac{1 \cdot 3 \cdots (2k-1)}{2^k (2k+1) k!} x^{2k+1}, \quad -1 < x < 1.
\]

11. Since \( R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1} \) we get for \( |x-a| < r \)

\[
|R_n(x)| \leq K \left( \frac{|x-a|}{r} \right)^{n+1} \to 0,
\]

hence the Taylor series for \( f \) at \( a \) converges to \( f \) on \((a-r, a+r)\).
15. We have $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and

$$\sum_{k=0}^{n} \frac{x^k y^{n-k}}{k! (n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \frac{(x+y)^n}{n!}.$$

Using Theorem 6.3.6 we get

$$e^x e^y = \left( \sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \left( \sum_{k=0}^{\infty} \frac{y^k}{k!} \right) = \sum_{n=0}^{\infty} \frac{(x+y)^n}{n!} = e^{x+y}.$$