7.3

4. The interior is
\[ \{(x, y) \in \mathbb{R}^2 : \| (x, y) \| < 1 \}, \]
the closure is
\[ \{(x, y) \in \mathbb{R}^2 : \| (x, y) \| \leq 1 \} \cup \{(x, y) \in \mathbb{R}^2 : y = 0, -2 \leq x \leq 2 \}, \]
and the boundary
\[ \{(x, y) \in \mathbb{R}^2 : \| (x, y) \| = 1 \} \cup \{(x, y) \in \mathbb{R}^2 : y = 0, -2 \leq x \leq 2 \}, \]

6. Since \( A \setminus B = A \cap B^c \) and \( B^c \) is open, it follows that \( A \setminus B \) is open. We don’t need \( B \subset A \).

Since \( B \setminus A = B \cap A^c \) and \( A^c \) is closed, it follows that \( B \setminus A \) is closed. We don’t need \( A \subset B \).

8. No. Let \( d = 1 \) and \( E = [0, 1] \cap \mathbb{Q} \). Then \( \overline{E}^c = (0, 1) \) but \( E^c = \emptyset \).

9. By double inclusion. Let \( x \in \overline{A \cup B} \). Then any ball \( B_r(x) \) intersects \( A \cup B \), so it intersects \( A \) or \( B \). It follows that \( x \in \overline{A} \cup \overline{B} \). Conversely, if \( x \in \overline{A} \cup \overline{B} \), then \( x \in \overline{A} \) or \( x \in \overline{B} \). Given a ball \( B_r(x) \), we have \( B_r(x) \cap A \neq \emptyset \) or \( B_r(x) \cap B \neq \emptyset \). In particular, \( B_r(x) \cap (A \cup B) \neq \emptyset \), so \( x \in \overline{A \cup B} \).

In general \( \overline{A \cap B} \neq \overline{A} \cap \overline{B} \). Let \( d = 1 \) and let \( A = \mathbb{Q}, B = \mathbb{R} \setminus \mathbb{Q} \). Then \( A \cap B = \emptyset \) and \( \overline{A} \cap \overline{B} = \emptyset \), but \( \overline{A} = \overline{B} = \mathbb{R} \), so \( \overline{A} \cap \overline{B} = \mathbb{R} \).

11. We will show that \( A^c \) is open. Let \( y \in A^c \). Since \( x \neq y \), we can find \( r > 0 \) such that \( B_r(x) \cap B_r(y) = \emptyset \). It suffices to take \( r < \frac{1}{2}\|x - y\| \). Since \( x_n \to x \), there is \( N \) such that \( x_n \in B_r(x) \) for all \( n \geq N \). By taking \( r \) so small that \( x_1, \ldots, x_{N-1} \) are not in \( B_r(y) \), we conclude that \( B_r(y) \subset A^c \).

12. Let \( x \in \overline{A} \). If \( x \in A \), we are done. If \( x \notin A \), for all \( k \geq 1 \) we can find \( x_{n_k} \in A \) with \( x_{n_k} \in B_{1/k}(x) \cap A \). It follows that \( x_{n_k} \to x \). Conversely, if \( x \) is the limit of a convergent subsequence, then \( x \in \overline{A} \) by Theorem 7.3.10. Notice that the points in \( A \) are limits of constant sequences.
7.4

1. Since $K$ is compact, the open cover $\{U_k\}_{k \geq 1}$ has a finite subcover $\{U_{k_1}, \ldots U_{k_p}\}$. Then $K \subset U_m$, where $m = \max\{k_1, \ldots, k_p\}$ since the sets are nested.

2. By contrapositive. Suppose $\left( \bigcap_{k} A_k \right) \cap K = \emptyset$. Then $K \subset \left( \bigcap_{k} A_k^c \right) = \bigcup_{k} A_k^c$ and $A_k^c$ are open. Since $K$ is compact, there is a finite subcover $\{A_{k_1}^c, \ldots, A_{k_p}^c\}$. Since $A_k$ are nested, $A_{k_1}^c \subset A_{k_2}^c \subset \cdots \subset A_{k_j}^c \subset \cdots$. It follows that there is $m$ with $K \subset A_m^c$. But this means $K \cap A_m = \emptyset$.

7. Since $K_2$ is closed and $K_1, K_2$ are disjoint, it follows that $K_1 \subset K_2^c$ and $K_2^c$ is open. By Theorem 7.4.10 we can find $V_1$ open with $K_1 \subset V_1 \subset K_2 \subset V_1^c$ and $V_1$ compact. In particular, $V_2 = V_1^c$ is open, $K_2 \subset V_2$ and $V_1 \cap V_2 = \emptyset$.

9. Let $K_1, K_2, \ldots, K_n$ be compact sets in $\mathbb{R}^d$. In particular, they are closed and bounded and also $K = K_1 \cup \cdots \cup K_n$ is closed and bounded. By the Heine-Borel Theorem, $K$ is compact.

If $d = 1$ and $K_n = [n, n+1]$ for $n \geq 1$, then each $K_n$ is compact. But $\bigcup_{n=1}^\infty K_n = [1, \infty)$ is not bounded, hence not compact.

10. Let $A, B$ be compact in a metric space $(X, \delta)$. Consider an open cover $\{U_i\}_{i \in I}$ of $A \cup B$. In particular, this is an open cover for $A$ and also an open cover for $B$. Since $A$ is compact, we can find a finite subcover $\{U_{i_1}, \ldots, U_{i_p}\}$. Similarly, since $B$ is compact, we can find a finite subcover $\{U_{j_1}, \ldots, U_{j_q}\}$. By putting together these open subcovers, we get a finite subcover of $A \cup B$.

If $A \cap B = \emptyset$, there is nothing to prove. To show that $A \cap B \neq \emptyset$ is compact, first notice that in a metric space any compact set is closed. Just adapt the proof of Theorem 7.4.5 by using the balls defined by the metric $\delta$. Given an open cover $\{U_i\}_{i \in I}$ of $A \cap B$, since $A \subset (A \cap B) \cup B^c$ and $B^c$ is open, by adding $B^c$ to $\{U_i\}_{i \in I}$ we get an open cover of $A$. Since $A$ is compact, we can find a finite subcover. By discarding $B^c$, we get a finite subcover of $A \cap B$. 