1. Find the Taylor series around $a = 2$ for

(a) $f(x) = \ln x$; (b) $g(x) = e^{3x}$.

a) For $k \geq 1$ we have $f^{(k)}(x) = (-1)^{k-1} \frac{(k-1)!}{x^k}$. Since $f(2) = \ln 2$, we get

$$\ln 2 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(x-2)^k}{k2^k}.$$ 

b) Since $g^{(k)}(x) = 3^k e^{3x}$ for all $k \geq 0$, we get

$$\sum_{k=0}^{\infty} \frac{3^k}{k!} (x-2)^k.$$ 

2. Find the closure, the interior, and the boundary for the sets

$A = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_{\infty} < 1\} \setminus \mathbb{Q}^2,$

$B = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_1 \leq 3, \text{ and } -1 < y \leq 2\},$

$C = \{(x, y) \in \mathbb{R}^2 : y = \sin \frac{1}{x}, x \neq 0\} \cup \{(0, 0)\}.$

The set $A$ is made of points in a square with at least one coordinate irrational. We cannot fit an open disk inside, so $A^o = \emptyset$. We get

$$\overline{A} = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_{\infty} \leq 1\}$$

since we can approximate rational numbers with irrationals. Also

$$\partial A = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_{\infty} \leq 1\}.$$ 

We have

$$\overline{B} = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_1 \leq 3, \text{ and } -1 \leq y \leq 2\},$$

$$B^o = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_1 < 3, \text{ and } -1 < y < 2\}$$

and

$$\partial B = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_1 \leq 3, \text{ and } y = -1 \text{ or } y = 2\} \cup \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_1 = 3, \text{ and } -1 \leq y \leq 2\}.$$
Since the graph of \( y = \sin \frac{1}{x} \) approaches the \( y \)-axis, we have

\[
C = \{ (x, y) \in \mathbb{R}^2 : y = \sin \frac{1}{x}, x \neq 0 \} \cup \{ (0, y) : -1 \leq y \leq 1 \}.
\]

We can not fit a disk inside \( C \), so \( C^\circ = \emptyset \) and

\[
\partial C = \{ (x, y) \in \mathbb{R}^2 : y = \sin \frac{1}{x}, x \neq 0 \} \cup \{ (0, y) : -1 \leq y \leq 1 \}.
\]

3. Consider vectors \( x, y \in \mathbb{R}^d \) with \( \|x\| = \|y\| = 1 \). Use the triangle inequality for the Euclidean norm to prove that if \( \|x + y\| = 2 \), then \( x = y \).

Since in the triangle inequality for the Euclidean norm we have equality \( \|x+y\| = \|x\| + \|y\| \) we get \( y = cx \) with \( |c| = 1 \). It follows that \( c = 1 \) and \( x = y \).

4. Find the interval of convergence for the power series

\[
a) \sum_{k=1}^{\infty} \frac{(-3)^k x^k}{k}, \quad b) \sum_{k=2}^{\infty} \frac{(x+1)^k}{k(\ln k)^3}.
\]

a) By the ratio test, \( R = 1/3 \). For \( x = -1/3 \) we get \( \sum_{k=1}^{\infty} \frac{1}{k} = \infty \) and for \( x = 1/3 \) we get

\[
\sum_{k=1}^{\infty} \frac{(-1)^k}{k}
\]

convergent by AST. Interval \((-1/3, 1/3]\).

b) By the ratio test, \( R = 1 \), hence \( |x+1| < 1 \) gives \(-2 < x < 0\).

For \( x = 0 \) we get \( \sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3} \), convergent by the integral test.

For \( x = -2 \), \( \sum_{k=2}^{\infty} \frac{(-1)^k}{k(\ln k)^3} \) is convergent by AST. Interval \([-2, 0]\).

5. Let \( f(x) = \sum_{k=1}^{\infty} \frac{\cos(kx)}{k^2} \). Prove that \( f \) is continuous on \( \mathbb{R} \) and that

\[
\int_{0}^{\pi/2} f(x)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3}.
\]
Since for all \( x \in \mathbb{R} \) we have \(|\cos(kx)| \leq 1\) and \( \sum_{k=1}^{\infty} \frac{1}{k^2} \) is convergent, by the Weierstrass \( M \)-test, the series of functions is uniformly convergent on \( \mathbb{R} \) and \( f \) is continuous. We have

\[
\int_0^{\pi/2} \cos(kx) \, dx = \frac{1}{k} \sin(kx) \bigg|_0^{\pi/2} = \frac{1}{k} \sin \frac{k\pi}{2}
\]

which is 0 for \( k \) even and \((-1)^p\) for \( k = 2p + 1 \). We get

\[
\int_0^{\pi/2} f(x) \, dx = \sum_{p=0}^{\infty} \frac{(-1)^p}{(2p+1)^3}.
\]

6. Find the sum of the series

\[
a) \sum_{k=1}^{\infty} \frac{k+1}{3^k}, \quad b) \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \cdot 4^k.
\]

a) Let \( f(x) = \sum_{k=1}^{\infty} (k+1)x^k \). Then

\[
f(x) = \left( \sum_{k=1}^{\infty} x^{k+1} \right)' = \left( \frac{x^2}{1-x} \right)' = \frac{x(2-x)}{(1-x)^2}
\]

and \( \sum_{k=1}^{\infty} \frac{k+1}{3^k} = f(1/3) = \frac{5}{4} \).

b) \( \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \cdot 4^k = -\frac{1}{4} \sum_{k=0}^{\infty} \frac{(-4)^{k+1}}{(k+1)!} = -\frac{1}{4} \sum_{k=1}^{\infty} \frac{(-4)^k}{k!} = -\frac{1}{4} (e^{-4} - 1) \).

7. Let \( A \subset \mathbb{R}^d \) be compact, and let \( B \subset A \) be closed. Use the definition with open covers to prove that \( B \) is also compact.

Let \( \{U_i\}_{i \in I} \) be an open cover of \( B \). Since \( A \) is compact, \( B^c \) is open and \( A \subset \bigcup_{i \in I} U_i \cup B^c \), we can find a finite subcover \( U_{i_1}, ..., U_{i_k}, B^c \) for \( A \). It follows that \( U_{i_1}, ..., U_{i_k} \) is a finite subcover for \( B \), hence \( B \) is compact.