Spectral analysis

Goals: Learn how to apply basic spectral analysis methods.

programs necessary for this Lab are available on the course Web site http://unr.edu/homepage/zal/STAT758.htm
1 Spectral analysis

Recall that the main tool of spectral analysis is the spectral density function (or just density) \( f(w) \) that is the Fourier transform of the process autocorrelation function \( \gamma(k) \):

\[
f(w) = \frac{1}{\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-iwk}.
\]

Using the symmetry of the acf \( \gamma(k) = \gamma(-k) \) we obtain

\[
f(w) = \frac{1}{\pi} \left[ \gamma(0) + 2 \sum_{k=1}^{\infty} \gamma(k) \cos wk \right].
\]

1.1 Periodogram

The estimation of \( f(w) \) is done using the periodogram of the time series \( x_t \) observed at the grid \( t = 1, \ldots, N \):

\[
I(w) = \frac{1}{\pi N} \left\| \sum_{t=1}^{N} x_t e^{iwt} \right\|^2.
\]

In order to obtain a good estimate of the spectrum \( f(w) \), the values of periodogram should be smoothed. The periodogram calculation and smoothing can be performed in R using the command

\[
> \text{sp<-spectrum(x, spans=3)}
\]

Here the option `spans` specify the extent to which the original periodogram values should be smoothed. (See other possible options in command help.)

1.2 Spectral techniques

Methods of spectral analysis can be used for various specific goals and often are more convenient than analysis in the time domain. Here we consider spectral smoothing methods. In terms of spectral analysis, smoothing means that we want to emphasize the low frequencies and deemphasize high frequencies in the frequency representation of the time series:

\[
X_t = \int_{0}^{\pi} \cos wt \; du(w) + \int_{0}^{\pi} \sin wt \; dv(w),
\]

where \( u(w) \) and \( v(w) \) are some uncorrelated processes with orthogonal increments.
To implement this program, we calculate the Fourier transformation $F_X(w)$ (do not confuse it with spectral distribution function) of the time series $X_t$, multiply it by a specifically designed filter $h(w)$, and then calculate inverse Fourier transformation to obtain a modified time series $Y_t$:

$$
Y_t = \frac{2}{\pi} \int_0^\infty e^{-iw} \left[ h(w) \int_{-\infty}^\infty X_s e^{iws} ds \right] dw \\
= \frac{2}{\pi} \int_{-\infty}^\infty X_s \left[ \int_0^\infty e^{-iw(t-s)} h(w) dw \right] ds. \quad (1)
$$

It is readily seen that if $h(w) = 1$, then $Y_t \equiv X_t$. In general, varying the shape of $h(w)$ allows us to select specific parts of the spectral representation of $X_t$.

In R we use the following commands to calculate Fourier and inverse Fourier transformations:

> f<-fft(x)

> f<-fft(x, inverse=T)

(Specific choices of $h(w)$ will be discussed and illustrated in class.)

**Assignment 1:** Estimate spectrum $f(w)$ of AR(1) model. Compare with the theoretical spectrum.

**Assignment 2:** Simulate ARMA(1,1,1) process with $\alpha = 0.3$, $\beta = 0.5$. Smooth it using a low-pass filter of your choice.