Problem 1. Find Taylor series for \( f(x) = e^{-x^2/2} \) based at \( b = 0 \).
(a) Write it in sigma notation.
(b) Write the first four nonzero terms.
(c) Find its interval of convergence.

Solution. (a) Plug in \( t = -x^2/2 \) into
\[
e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}
\]
and get
\[
f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{x^2}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n}.
\]
(b) \[
f(x) = \frac{(-1)^0}{2^0 0!}x^0 + \frac{(-1)^1}{2^1 1!}x^2 + \frac{(-1)^2}{2^2 2!}x^4 + \frac{(-1)^3}{2^3 3!}x^6 + \ldots = 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \ldots
\]
(c) The series for \( e^t \) converges everywhere, so the series for \( f(x) \) also converges everywhere, i.e. for all \( x \in \mathbb{R} = (-\infty, \infty) \).
**Problem 2.** Find the angle between the curves

\[ x = t, \ y = t, \ z = t^2 \] and \[ x = 0, \ y = s, \ z = s^2 - s. \]

**Solution.** First, let us find their point of intersection. Solve the system of equation:

\[ t = 0, \ t = s, \ t^2 = s^2 - s. \]

Obviously, the unique solution is \( t = s = 0 \). Let \( r_1(t) = < t, t, t^2 >, \ r_2(s) = < 0, s, s^2 - s >. \) Then \( r_1'(t) = < 1, 1, 2t >, \ r_2'(s) = < 0, 1, 2s - 1 >. \) Therefore, \( \mathbf{a} = r_1'(0) = < 1, 1, 0 > \) and \( \mathbf{b} = r_2'(0) = < 0, 1, -1 >. \) The angle \( \theta \) between the curves is the angle between these two vectors. Therefore,

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{||\mathbf{a}|| ||\mathbf{b}||} = \frac{1}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \pi/3.
\]
Problem 3. Let $O = (0, 0), \ A = (1, 0), \ B = (0, 1)$. Denote by $D$ the triangle $OAB$. Find

$$\iint_D y \, dA.$$ 

Solution. $D = \{0 \leq x \leq 1, \ 0 \leq y \leq 1 - x\}$. Indeed, the line $AB$ is given by $y = 1 - x$. That’s why

$$\iint_D y \, dA = \int_0^1 \left[ \int_0^{1-x} y \, dy \right] \, dx = \int_0^1 \frac{y^2}{2} \bigg|_{y=0}^{y=1-x} \, dx = \int_0^1 \frac{(1 - x)^2}{2} \, dx = \frac{1}{2} \int_0^1 (1 - x)^2 \, dx = \frac{1}{2} \int_0^1 u^2 \, du = \frac{1}{2} \cdot \frac{u^3}{3} \bigg|_{u=0}^{u=1} = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$
Problem 4. Find some $n$ such that for

$$x \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right], \ f(x) = \cos x, \ b = 0$$

we have:

$$|f(x) - T_n(x)| \leq 0.1.$$

Solution. We must ensure that

$$\frac{a^{n+1}}{(n + 1)!} M_{n+1} \leq 0.1,$$

where we denote

$$a = \frac{\pi}{2}, \ M_{n+1} := \max_{x \in [-\pi/2, \pi/2]} |f^{(n+1)}(x)|.$$

Note that all derivatives of $f(x)$ are either $\pm \sin x$ or $\pm \cos x$, and $|\pm \cos x|, |\pm \sin x| \leq 1$, so $M_{n+1} \leq 1$. That’s why we need

$$\frac{(\pi/2)^{n+1}}{(n + 1)!} \leq 0.1.$$

Trial-and-error:

$$n = 1 \Rightarrow \frac{(\pi/2)^2}{2!} \approx 1.234,$$

$$n = 2 \Rightarrow \frac{(\pi/2)^3}{3!} \approx 0.646,$$

$$n = 3 \Rightarrow \frac{(\pi/2)^4}{4!} \approx 0.254,$$

$$n = 4 \Rightarrow \frac{(\pi/2)^5}{5!} \approx 0.080,$$

so $n = 4$ works.
Problem 5. Find Taylor series for \( f(x) = \log x \) based at \( b = 4 \).
(a) Write it in sigma notation.
(b) Write the first four nonzero terms.
(c) Find its interval of convergence.

Solution. (a) Let \( y = x - 4 \). Let us find Taylor series for \( f(y + 4) = \log(2 + 4) \) based at \( b = 0 \).

\[
\begin{align*}
\log(4 + y) &= \log \left(1 + \frac{y}{4}\right) + \log 4 = \log 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{y}{4}\right)^n = \log 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} y^n.
\end{align*}
\]

Therefore,

\[
f(x) = \log 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n4^n} (x - 4)^n.
\]

(b)

\[
f(x) = \log 4 + \frac{(-1)^2}{14^1} (x-4) + \frac{(-1)^3}{24^2} (x-4)^2 + \frac{(-1)^3}{34^3} (x-4)^3 + \ldots = \log 4 + \frac{1}{8}(x-4) - \frac{1}{32} (x-4)^2 + \frac{1}{192} (x-4)^3 - \ldots
\]

(c) The interval of convergence for \( \log(1 + u) \) is \((-1, 1)\); therefore, the interval of convergence for this series is \(-1 < y/4 < 1 \Leftrightarrow -4 < y < 4 \Leftrightarrow -4 < x - 4 < 4 \Leftrightarrow 0 < x < 8 \), i.e. \((0, 8)\).