Double Integrals in Polar Coordinates
Lecture 15. 07/20/2012

Can we change variables in double integrals in the same way as for single-variable integrals? Yes, we can, but the general formula is quite cumbersome [Math 324]; now we shall study the most important change of variables: from Cartesian to polar coordinates: \( x = r \cos \theta, \ y = r \sin \theta \). We should rewrite \( D \) and \( f \) in polar coordinates.

Say, for \( f = \sqrt{x^2 + y^2}, \ D = \{ x^2 + y^2 \leq 1 \} \), we have: \( f = r, \ D = \{ 0 \leq \theta \leq 2\pi, \ 0 \leq r \leq 1 \} \).

(Do not forget: \( r \geq 0 \) always, and if there are no restrictions on \( \theta \), then set \( 0 \leq \theta \leq 2\pi \)) So, we can write

\[
\int \int_D f(x,y)\,dA = \int_0^{2\pi} \int_0^1 r\,dr\,d\theta.
\]

But this is wrong! Actually, we must add multiple \( r \) inside the integral: \( r^2\,dr\,d\theta \). This multiple \( r \) plays the role of, say, \( 2r \) in the change of variable \( u = x^2, \ du = 2xdx \) for single-variable integrals. Why is this? Because the small element of \( D \) has area \( r\,dr\,d\theta \), not \( dr\,d\theta \). (While in Cartesian coordinates it has area \( dx\,dy \).)

Here, the role of rectangles is played by polar rectangles: \( \{ \alpha \leq \theta \leq \beta, \ R_1 \leq r \leq R_2 \} \). And the area \( \Delta A \) of small polar rectangle

\[
\{ \theta_0 \leq \theta \leq \theta_0 + \Delta \theta, \ R \leq r \leq \Delta R + \Delta R \}
\]
is approximately \( R\Delta R\Delta \theta, \) not \( \Delta R\Delta \theta \). Indeed, its area is proportional to the angle width \( \Delta \theta \), so \( \Delta A = \frac{\Delta \theta}{2\pi} S \), where \( S \) is the area of the ring enclosed between the two circles with radii \( R \) and \( R + \Delta R \) (centered at the origin). But

\[
S = \pi(R + \Delta R)^2 - \pi R^2 = 2\pi R\Delta R + \pi(\Delta R)^2 \approx 2\pi R\Delta R,
\]
because \( \Delta R \) is small and \( (\Delta R)^2 \) is really small. Thus, \( \Delta A = R\Delta R\Delta \theta \).

Let us finish this example:

\[
\int \int_D f(x,y)\,dA = \int_0^{2\pi} \int_0^1 r^2\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r^2\,dr = \frac{2\pi}{3}.
\]

More generally, for \( D = \{ \alpha \leq \theta \leq \beta, \ r_1(\theta) \leq r \leq r_2(\theta) \} \)

\[
\int \int_D f(x,y)\,dA = \int_\alpha^\beta \int_{r_1(\theta)}^{r_2(\theta)} f(r \cos \theta, r \sin \theta)\,r\,dr\,d\theta.
\]

We can symbolically represent this as \( dA = dx\,dy = r\,dr\,d\theta \).

**Example.** The area of the disc with radius \( R \) is \( \pi R^2 \). The volume of the ball with radius \( R \) is \((4\pi/3)R^3 \). What is the 4D volume of the 4D ball \( \{ x^2 + y^2 + z^2 + w^2 \leq R^2 \} \) of radius \( R \)?

**Solution.** It is \( cR^4 \), where \( c \) is the volume of the unit ball \( B = \{ x^2 + y^2 + z^2 + w^2 \leq 1 \} \). For any \( (x, y, z, w) \in B \), we have: \( x^2 + y^2 \leq x^2 + y^2 + z^2 + w^2 \leq 1 \). Fix any \( x, y \) such that \( x^2 + y^2 \leq 1 \).

The section of \( B \) is the disc \( \{ z^2 + w^2 \leq 1 - x^2 - y^2 \} \) of radius \( \sqrt{1 - x^2 - y^2} \) centered at the origin.

Its area is \( \pi(\sqrt{1 - x^2 - y^2})^2 = \pi(1 - x^2 - y^2) \). So

\[
c = \int \int \pi(1 - x^2 - y^2)\,dA = \pi \int_0^{2\pi} \int_0^1 (1 - r^2)\,r\,dr\,d\theta = \pi \int_0^{2\pi} d\theta \int_0^1 (r - r^3)\,dr = \frac{2\pi}{3}.
\]
Thus, \( c = \frac{\pi^2}{2} \). The 4D volume of the 4D ball with radius \( R \) is \( \left( \frac{\pi^2}{2} \right) R^4 \).

**Example.** [Final Exam, Winter 2009, 8] Let

\[ D = \{(x, y) \in \mathbb{R}^2 \mid 4 \leq x^2 + y^2 \leq 4x, \ y \geq 0\} \]

(a) Draw a careful picture of \( D \).
(b) Compute the area of \( D \).

**Solution.**
(a) \( \{4 \leq x^2 + y^2\} \) is the exterior of the circle with radius 2 centered at the origin. \( \{x^2 + y^2 \leq 4x\} = \{(x-2)^2 + y^2 \leq 4\} \) is the interior of the circle centered at \( (2, 0) \) with radius 2. So this is the intersection of these two domains.

(b) In polar coordinates, \( x^2 + y^2 = r^2 \), \( x = r \cos \theta \), and let us find \( D \) in polar coordinates.

\[
4 \leq x^2 + y^2 \iff r \geq 2; \quad x^2 + y^2 \leq 4x \iff r^2 \leq 4r \cos \theta \iff r \leq 4 \cos \theta.
\]

So the limits on \( r \):

\[
2 \leq r \leq 4 \cos \theta.
\]

Also, \( y = r \sin \theta \geq 0 \iff \sin \theta \geq 0 \). Let us find limits on \( \theta \): \( 2 \leq r \leq 4 \cos \theta \Rightarrow \cos \theta \geq 1/2 \). So we have (draw the unit circle!):

\[
\cos \theta \geq 1/2, \quad \sin \theta \geq 0 \iff 0 \leq \theta \leq \pi/3.
\]

Finally, \( D = \{0 \leq \theta \leq \pi/3, \ 2 \leq r \leq 4 \cos \theta\} \).

The area of this domain is

\[
\int_0^{\pi/3} \int_2^{4 \cos \theta} 1 \, dr \, d\theta = \int_0^{\pi/3} \left[ \frac{r^2}{2} \right]_2^{4 \cos \theta} d\theta = \int_0^{\pi/3} \frac{1}{2} (16 \cos^2 \theta - 4) d\theta.
\]

But \( (16 \cos^2 \theta - 4)/2 = 8 \cos^2 \theta - 2 = 4(1 + \cos 2\theta) - 2 = 2 + 4 \cos 2\theta \). Therefore, this integral is equal to

\[
(2 \sin 2\theta + 2\theta)|^{\theta=\pi/3}_{\theta=0} = 2 \sin(2\pi/3) - 2 \sin 0 + \frac{2\pi}{3} = \sqrt{3} + \frac{2\pi}{3}.
\]

**Remark.** If we had

\[
D = \{4 \leq x^2 + y^2 \leq 4x\},
\]

then we would have just \( \cos \theta \geq 1/2 \), i.e. \( -\pi/3 \leq \theta \leq \pi/3 \). Do not hesitate to make \( \theta \) negative! Just be sure it does not overlap, i.e. the interval for \( \theta \) is not wider than \( 2\pi \). Writing \( -\pi/2 \leq \theta \leq 2\pi \) is wrong!

If you do not have any conditions on \( \theta \) (as in the 4D ball example), then simply let \( 0 \leq \theta \leq 2\pi \).