General Taylor Polynomials
Lecture 19. 08/01/2012

Recall: after integrating by parts twice, we get:

\[ f(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)dt = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{1}{2} \int_0^x (x-t)^2f'''(t)dt. \]

Integrating by parts once more, we get:

\[ f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{1}{6} \int_0^x (x-t)^3f^{(4)}(t)dt, \]

etc. Recall that \( f^{(k)} \) is the \( k \)th order derivative of \( f \). So for any \( n \) we get:

\[ f(x) = T_n(x) + \frac{1}{n!} \int_0^x (x-t)^n f^{(n+1)}(t)dt, \quad T_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n. \]

This polynomial \( T_n \) is called \( n \)th Taylor polynomial, or Taylor polynomial of degree \( n \), or \( n \)th order Taylor approximation. The integral is the error.

Here, \( n! = 1 \cdot 2 \cdot 3 \ldots \cdot n \) is called the factorial of \( n \). E.g. \( 1! = 1, 2! = 2, 3! = 6, 4! = 24, 5! = 120, 6! = 720 \), etc. We always have: \((n-1)! \cdot n = n!\) for \( n \geq 2 \). Plug in \( n = 1 \): \( 0! \cdot 1 = 1 \), so it is reasonable to accept a convention that \( 0! = 1 \) (although the product of zero factors does not make sense).

Let us find an error bound. Suppose that \( x > 0 \) (the case \( x < 0 \) is similar) and let \( M_{n+1} = \max_{0 \leq t \leq x} |f^{(n+1)}(t)| \). Then we have:

\[ \frac{M_{n+1}}{n!} \int_0^x (x-t)^n dt = \frac{M_{n+1}}{n!} \int_0^x u^n du = \frac{M_{n+1}}{n!(n+1)} x^{n+1} = \frac{|x|^{n+1}}{(n+1)!} M_{n+1}. \]

The same bound is valid for \( x < 0 \), with the following amendment: the maximum is taken over \( x \leq t \leq 0 \).

**Example.** \( f(x) = e^x \). All derivatives are the same, they are equal to \( e^x \), so \( f^{(k)}(0) = 1 \) for all \( k \). Therefore,

\[ T_n(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots + \frac{x^n}{n!}. \]

For \( x = 0.1 \) we have: \( M_{n+1} = \max_{0 \leq t \leq 0.1} e^t = e^{0.1} \leq 2 \), so the error bound is

\[ \frac{2 \cdot 0.1^{n+1}}{(n+1)!}. \]

For \( n = 1, 2 \) we already know this: it is 0.01 for \( n = 1 \) and 0.00033 for \( n = 2 \). For \( n = 3 \), this is \( 10^{-4}/12 < 10^{-5} \) - much less!

The larger \( n \) is, the better the approximation is. If you fix \( x \), then \( |f(x) - T_n(x)| \) gets smaller and smaller. Suppose you fixed some small \( \varepsilon > 0 \) and you want to find \( a > 0 \) such that for all \( x \in [-a, a] \) you have \( |f(x) - T_n(x)| \leq \varepsilon \). Then you need

\[ \frac{a^{n+1}}{(n+1)!} \max_{-a \leq t \leq a} |f^{(n+1)}(t)| \leq \varepsilon. \]
Example. $f(x) = \sin x$, $n = 4$, $\varepsilon = 0.01$. Then $f(0) = 0$, $f'(0) = \cos x$, $f''(0) = 1$, $f'''(x) = -\sin x$, $f''(0) = 0$, $f''(x) = -\cos x$, $f'''(0) = -1$, $f^{(4)}(x) = \sin x$, $f^{(4)}(0) = 0$. So

$$f(x) = \sin x \approx T_4(x) = x - \frac{x^3}{6}.$$ 

Since $f^{(5)}(t) = \cos t$, we need:

$$\frac{a^5}{5!} \max_{-a \leq t \leq a} |\cos t| \leq 0.01.$$ 

This max is equal to 1 (for all $t$ we have $|\cos t| \leq 1$, while $|\cos 0| = 1$). So we need $a^5/5! \leq 0.01$. If we let $a^5/5! = 0.01 \Leftrightarrow a^5/120 = 0.01 \Leftrightarrow a^5 = 1.2 \Leftrightarrow a = \sqrt[5]{1.2} \geq 1$, so $a = 1$ is valid. (Remember, we do not need to find the best, i.e. the largest possible $a$.) Recall that for $T_1$ we had $a = 0.271$, and for $T_2$ we had $a = 0.391$. The more terms we take, the more precise this approximation is.

Example. For $f(x) = \sin x$, find $n$ such that $|f(x) - T_n(x)| \leq 10^{-4}$ for all $-1 \leq x \leq 1$. We need to find $n$ such that for $a = 1$ we have:

$$\frac{a^{n+1}}{(n + 1)!} \max_{-a \leq t \leq a} |f^{(n+1)}(t)| \leq 10^{-4}.$$ 

The derivative of any order of $f$ is either $\pm \sin t$ or $\pm \cos t$; anyway, $|f^{(n+1)}(t)| \leq 1$, so this max is dominated by 1. So we need: $a^{n+1}/(n + 1)! \leq 10^{-4}$. Plug in $a = 1$: $1/(n + 1)! \leq 10^{-4} \Leftrightarrow (n + 1)! \geq 10^4 = 10000$. Try some $n$: for $n = 5$ we have $(n + 1)! = 6! = 720 < 10000$; for $n = 6$ we have $(n + 1)! = 7! = 5040 < 10000$, and for $n = 7$ we have $(n + 1)! = 8! = 40320 > 10000$. So $n = 7$ is a correct answer (of course, larger values of $n$ are also answers). And

$$f(x) = \sin x \approx T_7(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040}.$$ 

Finally, if we approximate $f(x)$ for $x \approx b$, then

$$f(x) = T_n(x) + \frac{1}{n!} \int_b^x (x - t)^n f^{(n+1)}(t)dt,$$

$$T_n(x) = f(b) + f'(b)(x - b) + \frac{f''(b)}{2} (x - b)^2 + \frac{f'''(b)}{6} (x - b)^3 + \ldots + \frac{f^{(n)}(b)}{n!} (x - b)^n.$$ 

And the error bound is

$$\frac{|x - b|^{n+1}}{(n + 1)!} M_{n+1}, \quad M_{n+1} := \max |f^{(n+1)}(t)|,$$

where the maximum is taken over $t$ between $b$ and $x$. 

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