Vectors in Three Dimensions
Lecture 2. 06/19/2012

Definition of vectors. The term vector is used to indicate a quantity (e.g. velocity or force) with both magnitude and direction. A quantity which has only magnitude without direction (i.e. real numbers) is called a scalar. We denote a vector by printing a letter in boldface: \( \mathbf{v} \) or by an arrow: \( \vec{v} \).

Suppose a particle has moved from the point \( A \) to the point \( B \). The displacement vector \( \mathbf{v} = \mathbf{AB} \) has initial point \( A \) and terminal point \( B \). When we write a vector as a combination of two letters, the first letter denotes the initial point and the second letter denotes the terminal point.

If two vectors \( \mathbf{u} \) and \( \mathbf{v} \) have the same direction and length, they are called equal: \( \mathbf{u} = \mathbf{v} \). The zero vector \( \mathbf{0} = \mathbf{AA} \) has length 0 and no specific direction.

The magnitude, or length, of the vector \( \mathbf{v} = \mathbf{AB} \) is the distance from \( A \) to \( B \). It is denoted by \( |\mathbf{v}| \).

Addition. If we have two vectors \( \mathbf{u} \) and \( \mathbf{v} \), how to add them?

Triangle Rule. Suppose a particle moved from \( A \) to \( B \) so that \( \mathbf{u} = \mathbf{AB} \) and then it moved from \( B \) to \( C \) so that \( \mathbf{v} = \mathbf{BC} \). Then the "sum" of these movements is the movement from \( A \) to \( C \), so \( \mathbf{u} + \mathbf{v} = \mathbf{AC} \) by definition.

Parallelogramm Rule. Draw \( \mathbf{u} = \mathbf{AB} \) and \( \mathbf{v} = \mathbf{AD} \) from the same initial point \( A \) and construct a parallelogramm based on these vectors: a parallelogramm \( ABCD \). Then, by definition, \( \mathbf{AC} = \mathbf{u} + \mathbf{v} \).

These rules are equivalent, since \( \mathbf{BC} = \mathbf{AD} = \mathbf{v} \).

Multiplication by a scalar. The idea is: \( 2\mathbf{v} = \mathbf{v} + \mathbf{v} \) has the same direction and its magnitude is the doubled magnitude of \( \mathbf{v} \).

In general, let \( c \) be a scalar and let \( \mathbf{v} \) be a vector. Then \( \mathbf{u} = c\mathbf{v} \) has magnitude \( |\mathbf{u}| = |c||\mathbf{v}| \), and its direction is the same as the direction of \( \mathbf{v} \) if \( c > 0 \), and is opposite if \( c < 0 \). If \( c = 0 \), then \( c\mathbf{v} = \mathbf{0} \). Also, if \( \mathbf{v} = \mathbf{0} \), then \( c\mathbf{v} = \mathbf{0} \).

Important example: if \( \mathbf{AB} = \mathbf{v} \), then \( \mathbf{BA} = -\mathbf{v} = (-1)\cdot \mathbf{v} \). The magnitude of \( -\mathbf{v} \) is the same as the magnitude of \( \mathbf{v} \), and the length is opposite.

Subtraction. For any vectors \( \mathbf{u} \) and \( \mathbf{v} \), \( \mathbf{u} - \mathbf{v} = \mathbf{u} + (\mathbf{-v}) \). Draw the vectors \( \mathbf{u}, \mathbf{v} \) from the same initial point \( A \): \( \mathbf{u} = \mathbf{AB} \), \( \mathbf{v} = \mathbf{AC} \). Then \( -\mathbf{v} = \mathbf{CA} \) and \( \mathbf{u} - \mathbf{v} = \mathbf{AB} + \mathbf{CA} = \mathbf{CB} \).

Components. We can write every vector as three real numbers: suppose \( \mathbf{u} = \mathbf{OA} \), where \( O \) is the origin and \( A = (x_0, y_0, z_0) \) is a point in space. Then, by definition, \( \mathbf{u} = < x_0, y_0, z_0 > \). These numbers \( x_0, y_0, z_0 \) are called the components of \( \mathbf{u} \). So any vector is defined by its components, and, conversely, any three real numbers are components of some vector.

If \( \mathbf{u} = \mathbf{AB} \), where \( A = (x_1, y_1, z_1) \) and \( B = (x_2, y_2, z_2) \), then \( \mathbf{u} = < x_2 - x_1, y_2 - y_1, z_2 - z_1 > \). Indeed, suppose \( \mathbf{u} = < \alpha, \beta, \gamma > \). Then \( B = (x_1 + \alpha, y_1 + \beta, z_1 + \gamma) \), but at the same time \( B = (x_2, y_2, z_2) \). Comparing the coordinates of \( B \), we get:

\[
x_1 + \alpha = x_2, \quad y_1 + \beta = y_2, \quad z_1 + \gamma = z_2,
\]
so we get:

\[
\alpha = x_2 - x_1, \quad \beta = y_2 - y_1, \quad \gamma = z_2 - z_1.
\]

Example. If \( A = (0, -2, 3) \) and \( B = (1, -3, 6) \), then \( \mathbf{AB} =< 1, -1, 3 > \).
Applications of components. If \( \mathbf{u} = <u_1, u_2, u_3> \) and \( \mathbf{v} = <v_1, v_2, v_3> \), then \( \mathbf{u} + \mathbf{v} = <u_1 + v_1, u_2 + v_2, u_3 + v_3> \). Indeed, suppose \( \mathbf{u} = \overrightarrow{OA} \) and \( \mathbf{v} = \overrightarrow{AB} \). Then \( \overrightarrow{A} = (u_1, u_2, u_3) \), \( \overrightarrow{B} = (u_1 + v_1, u_2 + v_2, u_3 + v_3) \) and \( \mathbf{u} + \mathbf{v} = \overrightarrow{OB} = <u_1 + v_1, u_2 + v_2, u_3 + v_3> \).

Also, if \( \mathbf{u} = <u_1, u_2, u_3> \), then \( c\mathbf{u} = <cu_1, cu_2, cu_3> \). (See the book for detailed explanation.)

In particular, \( -\mathbf{u} = <-u_1, -u_2, -u_3> \). And \( \mathbf{v} - \mathbf{u} = \mathbf{v} + (-\mathbf{u}) = <v_1 - u_1, v_2 - u_2, v_3 - u_3> \).

Thus, to add or subtract vectors, we just add or subtract their components.

The magnitude \( |\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2} \). Indeed, let \( \mathbf{u} = \overrightarrow{OA} \). Then \( \overrightarrow{A} = (u_1, u_2, u_3) \). And the distance between \( O \) and \( A \) (which is the magnitude \( |\mathbf{u}| \) of \( \mathbf{u} \)) is \( \sqrt{u_1^2 + u_2^2 + u_3^2} \).

**Example.** For \( \mathbf{u} = <1, 2, 3> \), we have: \( |\mathbf{u}| = \sqrt{1 + 4 + 9} = \sqrt{14} = 3 \).

The vectors \( \mathbf{i} = <1, 0, 0> \), \( \mathbf{j} = <0, 1, 0> \) and \( \mathbf{k} = <0, 0, 1> \) play a special role: any vector can be represented as a linear combination of these three vectors. More precisely, if \( \mathbf{u} = <u_1, u_2, u_3> \), then \( \mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \).

A unit vector is a vector with length 1. E.g. \( \mathbf{i}, \mathbf{j}, \mathbf{k} \) are unit vectors. If \( \mathbf{a} \neq \mathbf{0} \), then \( u = \frac{1}{|\mathbf{a}|}\mathbf{a} \) is a unit vector which has the same direction as \( \mathbf{a} \), because its length is \( \frac{1}{|\mathbf{a}|}|\mathbf{a}| = 1 \). Say, the unit vector in the direction of \( \mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k} \) is \( 2/3\mathbf{i} - 1/3\mathbf{j} - 2/3\mathbf{k} \), because \( |\mathbf{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2} = 3 \).

Applications in Physics. The magnitude of a velocity is called speed. Suppose a wind is blowing from N45°W at 50 km/h. This means the direction is 45° west from the northern direction. A pilot is steering a plane in N60°E at an airspeed (speed in still air) 250 km/h. The true course, or track, is the direction of the resultant of the velocity vectors of the plane and the wind. The ground speed is the magnitude of the resultant.

Here, the velocity of the wind is

\[
\mathbf{u} = <50 \cos 45^\circ, -50 \sin 45^\circ> = <50/\sqrt{2}, -50/\sqrt{2}> = <25\sqrt{2}, -25\sqrt{2}>,
\]

and the airspeed is

\[
\mathbf{v} = <250 \sin 60^\circ, 250 \cos 60^\circ> = <250/2, 250 \cdot \sqrt{3}/2> = <125, 125\sqrt{3}>.
\]

So the resultant is

\[
\mathbf{w} = \mathbf{u} + \mathbf{v} = <25\sqrt{2} + 125, -25\sqrt{2} + 125\sqrt{3}> \approx <160, 181>.
\]

Thus, \( |\mathbf{w}| = \sqrt{160^2 + 181^2} = 242 \) - the ground speed, and the true course is N41°E, because 41° \( \approx \arctan(160/181) \).