Power Series and Taylor Series
Lecture 21. 08/06/2012

Power series. This is the series of the form \( \sum_{n=0}^{\infty} c_n x^n \), where \( c_n \) are some numbers, \( x \) is a variable. Assume the following limit exists:

\[
\lim_{n \to \infty} \left| \frac{c_n}{c_{n+1}} \right| = R.
\]

Apply the Ratio Test: \( a_n = c_n x^n \), so

\[
\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{|x|}{R}.
\]

So if \( |x| < R \), this series converges, and if \( |x| > R \), it diverges. The interval \((-R, R)\) is called the interval of convergence, and \( R \) is the radius of convergence. We cannot say anything about the points \( \pm R \): the series may converge or diverge.

Example. 1. \( \sum x^n / n! \): \( c_n = 1/n! \), so \( |c_n/c_{n+1}| = (1/n!)/(1/(n+1)!)) = (n+1)!/n! = n+1 \to \infty \), so \( R = \infty \), it converges for all real \( x \). If there are factorials in the denominator of \( c_n \), the interval of convergence is the whole real line. In fact, this series converges to \( e^x \).

2. \( \sum \frac{(-1)^n+1 x^n}{n} \): \( c_n = (-1)^n+1/n \), and \( |c_n/c_{n+1}| = (n+1)/n = 1+1/n \to 1 \), so \( R = 1 \). In fact, the sum is ln(1 + x). If \( c_n \) contains only powers of \( n \) in the numerator and/or denominator (possibly with alternating signs - it does not make a difference), then \( R = 1 \).

Taylor series. We have Taylor approximation for \( f(x) \), \( x \approx 0 \):

\[
f(x) \approx T_n(x) = f(0) + f'(0) x + \frac{f''(0)}{2} x^2 + \frac{f'''(0)}{6} x^3 + \ldots + \frac{f^{(n)}(0)}{n!} x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k.
\]

The larger \( n \) is, the better this approximation is, because the error bound usually becomes smaller and smaller and tends to zero as \( n \to \infty \). If we set \( n = \infty \), in most cases we will have exact equation:

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.
\]

This is called Taylor series for \( f(x) \) centered at 0. We can make it centered at \( b \):

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(b)}{n!} (x-b)^n.
\]

Standard Taylor series. Let us calculate such series for the following four easiest functions: \( 1/(1-x) \), \( e^x \), \( \sin x \), \( \cos x \).

1. \( f(x) = 1/(1-x) \). Actually, we already know this:

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.
\]
2. \( f(x) = e^x \). Then all derivatives are also \( e^x \), and their values at zero are 1. So
\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.
\]

3. \( f(x) = \sin x \). Then
- \( f^{(0)}(0) = f(0) = \sin 0 = 0 \);
- \( f^{(1)}(0) = f'(0) = \cos 0 = 1 \);
- \( f^{(2)}(0) = f''(0) = -\sin 0 = 0 \);
- \( f^{(3)}(0) = f'''(0) = -\cos 0 = -1 \);
- \( f^{(4)}(0) = \sin 0 = 0 \);
- \( f^{(5)}(0) = \cos 0 = 1 \);
- \( f^{(6)}(0) = -\sin 0 = 0 \);
- \( f^{(7)}(0) = -\cos 0 = -1 \), etc.

Therefore, the Taylor series has the form
\[
x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.
\]

We can write it in closed form. Indeed,
\[
f^{(n)}(0) = \begin{cases} 0, & n = 2k; \\
(-1)^k, & n = 2k + 1; 
\end{cases}
\]
so
\[
\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.
\]

4. \( f(x) = \cos x \). Then similarly
\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.
\]

The first series (geometric series) has the interval of convergence \((-1, 1)\), and the three other series (for \( e^x \), \( \cos x \), \( \sin x \)) converge for all \( x \).