Lines and Planes

Lecture 5. 06/25/2012

Lines. A line $l$ in $\mathbb{R}^3$ is defined by any point $P_0 = (x_0, y_0, z_0)$ on it and its direction, represented by a directional vector $\mathbf{v} = \langle a, b, c \rangle$. A point $P = (x, y, z)$ lies on $l$ iff $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r}_0 = \langle x - x_0, y - y_0, z - z_0 \rangle$ is parallel to $\mathbf{v}$, where $\mathbf{r} = \overrightarrow{OP}$, $\mathbf{r}_0 = \overrightarrow{OP}_0$, i.e. iff for some real number $t$ we have:

$$\mathbf{r} - \mathbf{r}_0 = t\mathbf{v}.$$ 

This is called the vector equation of $l$. The scalar $t$ is called the parameter. Rewrite this as

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$ 

These are parametric equations of $l$. Each value of $t$ gives a point on $l$, and vice versa.

Example. $P_0 = (1, 2, 3)$, $\mathbf{v} = \langle -1, 2, 0 \rangle$. Then we have:

$$x = 1 - t, \quad y = 2 + 2t, \quad z = 3.$$ 

Say, $t = 1 \Rightarrow (x, y, z) = (0, 4, 3), \quad t = -1 \Rightarrow (x, y, z) = (2, 0, 3)$. This line intersects the $xz$-plane at $2 + 2t = 0 \Rightarrow t = -1$, i.e. at the point $(2, 0, 3)$. We can change $P_0$ to any other point on $l$, i.e. $P_1(0, 4, 3)$; then we have:

$$x = -t, \quad y = 4 + 2t, \quad z = 3.$$ 

These are also parametric equations for $l$. Also, we can change $\mathbf{v}$ to any nonzero vector parallel to $\mathbf{v}$, say $-2\mathbf{v} = \langle 2, -4, 0 \rangle$, so that

$$x = 1 + 2t, \quad y = 2 - 4t, \quad z = 3.$$ 

Line segment. Suppose $P_0$ and $P_1$ are distinct points in $\mathbb{R}^3$, with $\overrightarrow{P_0P} = \mathbf{r}_0$, $\overrightarrow{P_1P} = \mathbf{r}_1$. The directional vector of the line $P_0P_1$ is $\mathbf{r}_1 - \mathbf{r}_0$, and the vector equation of this line is

$$\mathbf{r}(t) = \mathbf{r}_0 + t(\mathbf{r}_1 - \mathbf{r}_0).$$ 

The values $t = 0$ and $t = 1$ correspond to $P_0$ and $P_1$. The segment $P_0P_1$ is given by $0 \leq t \leq 1$.

Skewness. In $\mathbb{R}^2$, two lines either intersect or are parallel. In $\mathbb{R}^3$, there is another alternative: they are skew, i.e. do not lie on the same plane. Indeed, if they are parallel or intersect, then they lie on a common plane.

Example. Let $l_1$ be given by $x = 1 + t$, $y = -2 + 3t$, $z = 4 - t$. Let $l_2$ be given by $x = 2s$, $y = 3 + s$, $z = -3 + 4s$. They are not parallel because their directional vectors $\langle 1, 3, -1 \rangle$ and $\langle 2, 1, 4 \rangle$ are not parallel. (Their components are not proportional; also, you can verify this by taking their cross product and observing it is not zero.) Let us find their point of intersection. It satisfies the system of equations

$$1 + t = 2s, \quad -2 + 3t = 3 + s, \quad 4 - t = -3 + 4s.$$ 

We have: $t = 2s - 1$ from the first equation. Plug it into the second equation: $-2 + 3(2s - 1) = 3 + s, \quad 4 - (2s - 1) = -3 + 4s$, i.e. $6s - 5 = 3 + s, \quad 5 - 2s = -3 + 4s$, i.e. $5s = 8, \quad 6s = 8, \quad s = 8/5, \quad s = 4/3$. So this system does not have a solution, and the lines actually do not intersect, i.e. they are skew.
Planes. A plane does not have a "directional vector", since a single vector parallel to a plane is not enough to grasp its "direction". However, a normal vector, i.e. a nonzero vector orthogonal to the plane is enough. Let \( \mathbf{n} = \langle a, b, c \rangle \) be a normal vector, and let \( P_0 = (x_0, y_0, z_0) \) be any point on the plane \( \pi \). Then \( P = (x, y, z) \in \pi \) iff \( \overrightarrow{P_0P} \perp \mathbf{n} \iff \overrightarrow{P_0P} \cdot \mathbf{n} = 0 \). If \( \mathbf{r} = \overrightarrow{OP} \), \( \mathbf{r_0} = \overrightarrow{OP_0} \), then we have:

\[
(\mathbf{r} - \mathbf{r_0}) \cdot \mathbf{n} = 0 \iff \mathbf{r} \cdot \mathbf{n} = \mathbf{r_0} \cdot \mathbf{n}.
\]

This is a vector equation of \( \pi \). Rewrite it in coordinate form:

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.
\]

**Example.** \( \mathbf{n} = \langle -1, 2, 0 \rangle \), \( P_0 = (1, 2, 3) \), then we have:

\[
-1(x - 1) + 2(y - 2) + 0(z - 3) = 0 \iff -x + 1 + 2y - 4 = 0 \iff 2y - x = 3.
\]

We can also take a multiple of \( \mathbf{n} \), and change point \( P_0 \) to some other point on the plane, e.g. to \((0, 3/2, 1)\), and the equation will remain the same. Lack of the \( z \)-coordinate means this plane is parallel to the \( z \)-axis. Indeed, the directional vector of the \( z \)-axis, i.e. \( \mathbf{k} \), is orthogonal to \( \mathbf{n} \):

\[
\mathbf{k} \cdot \mathbf{n} = 0,
\]

and so \( \mathbf{k} \) is parallel to this plane.

The \( x \)-intercept (i.e. intersection with the \( x \)-axis): let \( y = z = 0 \), then \(-x = 3 \Rightarrow x = -3 \). The \( y \)-intercept: \( x = z = 0 \Rightarrow 2y = 3 \Rightarrow y = 3/2 \).

**Example.** The plane that passes through \( P = (0, 0, 0) \), \( Q = (1, -1, 0) \), \( R = (1, 0, 2) \) has a normal vector: \( \mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1, -1, 0 \rangle \times \langle 1, 0, 2 \rangle = \langle -2, -2, 1 \rangle \), and the equation is:

\[
(\cdot)(x - 0) + (-2)(y - 0) + 1(z - 0) = 0 \iff -2x - 2y + z = 0.
\]

**Example.** The line \( x = 1 - t \), \( y = 2 + 2t \), \( z = 3 \) intersects the plane \( 2y - x = 3 \) at \( 2(2 + 2t) - (1 - t) = 3 \iff 5t + 3 = 3 \iff t = 0 \), i.e. at \((x, y, z) = (1, 2, 3)\).

**Example.** The angle between the planes is defined as the acute angle between their normal vectors. Consider the planes \( x + y + z = 1 \) and \( y - 2z = 0 \). Let \( \mathbf{n_1} = \langle 1, 1, 1 \rangle \), \( \mathbf{n_2} = \langle 0, 1, -2 \rangle \). Then the angle \( \theta \) between them satisfies

\[
\cos \theta = \frac{\mathbf{n_1} \cdot \mathbf{n_2}}{\| \mathbf{n_1} \| \| \mathbf{n_2} \|} = \frac{-1}{\sqrt{3} \sqrt{5}} = -\frac{1}{\sqrt{15}}.
\]

This angle is obtuse, so let us take \( \theta' = \pi - \theta \); it has \( \cos \theta' = 1/\sqrt{15} \), \( \theta' = \arccos(1/\sqrt{15}) \). If we took \( \mathbf{n_2} = \langle 0, -1, 2 \rangle \), we would get \( \theta' \) immediately.

Let us find the line of intersection of these two planes. First, its directional vector lies on each of these planes; therefore, it is orthogonal to both normal vectors, and we can take their cross product as the directional vector. (Recall that the directional vector is defined up to a multiplication by a nonzero scalar.) So \( \mathbf{v} = \mathbf{n_1} \times \mathbf{n_2} = \langle -3, 2, 1 \rangle \).

Then, find any point on this line. The system of two equations with three variables has infinitely many solutions, because there are “too many” variables. Let one of the variables \( b \) equal to zero; then you will have equal number of equations and variables. E.g. let \( z = 0 \); then \( y = 0 \) and \( x = 1 \), so \( P_0(1, 0, 0) \) lies on this line, and its equation:

\[
x = 1 - 3t, \quad y = 2t, \quad z = t.
\]