Arc length. Here, \( \mathbf{r}(t) \) is a vector function, and \( v = |\mathbf{r}'| \) is the speed. The length of \( \mathbf{r}(t), \alpha \leq t \leq \beta \) (= the distance covered at time \([\alpha, \beta]\)), is given by \( \int_\alpha^\beta v(t)dt \). Indeed, split \([\alpha, \beta]\) into small subintervals \( \alpha = t_0 < t_1 < \ldots < t_n = \beta \). During the time \([t_{k-1}, t_k]\), the speed is approximately \( v(t_k) \), so the distance covered is \((t_k - t_{k-1})v(t_k)\). Sum up these distances and get:

\[
\sum_{k=1}^n v(t_k)(t_k - t_{k-1}) \approx \int_\alpha^\beta v(t)dt.
\]

Example. The length of \( x = \cos t, \ y = \sin t, \ z = t, \ 0 \leq t \leq \pi \):

\[
x' = -\sin t, \ y' = \cos t, \ z' = 1, \ v(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}.
\]

Thus, the length is \( \int_0^\pi \sqrt{2}dt = \sqrt{2}\pi \).

Reparametrization. Two vector functions \( x = t, y = 3t \) and \( x = s^3, y = 3s^3 \) give the same curve. In general, for any increasing function \( t = \varphi(s) \ \mathbf{R}(s) = \mathbf{r}(\varphi(s)) \) gives the same curve (probably its part). And \( \mathbf{R}'(s) = \mathbf{r}'(t)(dt/ds) \), so the new speed \( V = \left| \mathbf{R}' \right| = \left| \mathbf{r}' \right|(dt/ds) = v(dt/ds) \). We move along the same path, but with different speed. A natural reparametrization is

\[
s(t) = \int_0^t v(u)du \Rightarrow \frac{ds}{dt} = v \Rightarrow \frac{dt}{ds} = \frac{1}{v} \Rightarrow V = \frac{1}{v} = 1.
\]

So we move along the path with unit speed.

Example. For the curve above,

\[
s = \int_0^t \sqrt{2}du = \sqrt{2}t \Rightarrow x = \cos \frac{s}{\sqrt{2}}, \ y = \sin \frac{s}{\sqrt{2}}, \ z = \frac{s}{\sqrt{2}}.
\]

TNB Basis and curvature. The unit tangent vector is \( \mathbf{T} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{v}\mathbf{r}'(t) \). The acceleration \( \mathbf{a} = \mathbf{r}'' \) has two components: a tangential, which is parallel to \( \mathbf{T} \) and is equal to

\[
\mathbf{a}_r := \text{proj}_\mathbf{T} \mathbf{r}'' = \frac{\mathbf{r}'' \cdot \mathbf{T}}{|\mathbf{T}|^2} \mathbf{T} = (\mathbf{r}'' \cdot \mathbf{T}) \mathbf{T},
\]

and a normal, which is orthogonal to \( \mathbf{T} \) and is equal to

\[
\mathbf{a}_n := \mathbf{T} \times (\mathbf{r}'' \times \mathbf{T}) = \mathbf{r}''(\mathbf{T} \cdot \mathbf{T}) - \mathbf{T}(\mathbf{r}'' \cdot \mathbf{T}) = \mathbf{r}'' - \mathbf{a}_r.
\]

Note that

\[
\mathbf{r}'' \cdot \mathbf{T} = \frac{\mathbf{r}'' \cdot \mathbf{T}}{v} = \frac{\mathbf{r}' \cdot \mathbf{r}'' + \mathbf{r}'' \cdot \mathbf{r}'}{2v} = \frac{(\mathbf{r}' \cdot \mathbf{r}')'}{2v} = \frac{\left|\mathbf{r}'\right|^2}{2v} = \frac{v^2}{2v} = \frac{2vv'}{2v} = v'.
\]

So \( \mathbf{a}_r = v'\mathbf{T} \). It measures the change \( v' \) of speed (= magnitude of velocity). Since \( \mathbf{T} \) is orthogonal to \( \mathbf{r}' \times \mathbf{T} \), we have:

\[
|\mathbf{a}_n| = |\mathbf{T}|\left|\mathbf{r}'' \times \mathbf{T}\right| \sin(\pi/2) = |\mathbf{r}'' \times \mathbf{T}| = \frac{|\mathbf{r}' \times \mathbf{r}'|}{v}.
\]
Denote by $N = T'/|T'|$ the unit vector in the direction of $T'$. Let us calculate it. Note that

$$T' = \frac{r''v - r'v'}{v^2} = \frac{r''v - r'\frac{r''r'}{v}}{v^2} = \frac{r''}{v} - \frac{r'(r'' \cdot r')}{v^3} = \frac{r'' - T(r'' \cdot T)}{v} = \frac{a_n}{v}.$$ 

So $T'$ and $a_n$ measure the change of direction of velocity, i.e. how curved is this curve, how far is it from a straight line. We have: $T'|a_n$, and $a_n \perp T$. So $T' \perp T$, and $N \perp T$.

The curve near the given point $r$ moves near the plane parallel to $T$ and $N$ and passing through $r(t)$. This is called the osculating plane (="approximating plane"). A normal vector for the osculating plane is $B = T \times N$, a binormal vector. A normal plane is a plane that passes through the point $r(t)$ and is orthogonal to $T$; it is "orthogonal to the curve at the point $r(t)$".

The vectors $T$, $N$, $B$ are mutually orthogonal and have length 1. ($|B| = 1 \cdot 1 \cdot \sin(\pi/2) = 1$.)

The magnitude of $T'$ is not a perfect measure of curvature, since after reparametrization it can change (by a factor of $dt/ds$). If we divide it by $v$ (which also changes by this factor), the ratio will be invariant under parametrization, and so $k(t) = |T'|/v$ is called the curvature of this curve at point $r(t)$. Find it:

$$|T'| = \frac{1}{v}|a_n| = \frac{1}{v^2}|r' \times r''| \Rightarrow k(t) = \frac{|r' \times r''|}{v^3}.$$ 

Example. For the curve above, $v = \sqrt{2}$, so

$$T(t) = \frac{r'(t)}{v} = \langle -\frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \rangle \Rightarrow T'(t) = \langle -\frac{1}{\sqrt{2}} \cos t, -\frac{1}{\sqrt{2}} \sin t, 0 \rangle,$$

so $|T'| = 1/\sqrt{2}$ and $N = T'/|T'| = \sqrt{2}T' = \langle -\cos t, -\sin t, 0 \rangle$. Finally,

$$B = T \times N = \begin{vmatrix} i & j & k \\ -\sin t/\sqrt{2} & \cos t/\sqrt{2} & 1/\sqrt{2} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \langle \frac{1}{\sqrt{2}} \sin t, -\frac{1}{\sqrt{2}} \cos t, \frac{1}{\sqrt{2}} \rangle.$$

The curvature is $k(t) = |T'|/v = 1/2$, and $a = \langle -\cos t, -\sin t, 0 \rangle \perp r'$, so $a_n = a$, $a_T = 0$. The normal plane at $t = 0$, i.e. at the point $(\cos 0, \sin 0, 0) = (1, 0, 0)$, has normal vector $T(0) = \langle 0, 1/\sqrt{2}, 1/\sqrt{2} \rangle$, i.e. normal vector $< 0, 1, 1 >$, so its equation is $0(x - 1) + 1(y - 0) + 1(z - 0) = 0 \iff y + z = 0$. The osculating plane at $t = 0$ has normal vector $B(0) = \langle 0, -1/\sqrt{2}, 1/\sqrt{2} \rangle$, i.e. normal vector $< 0, -1, 1 >$, so its equation is $0(x - 1) - 1(y - 0) + 1(z - 0) = 0 \iff -y + z = 0$. 

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