Problem 1. [Milakis, Spring 2007, 1] Let $a = <1, 1, x>$ and $b = <0, x, 1>$. 
(a) Find $x \in \mathbb{R}$ such that $a$ and $b$ are orthogonal.
(b) Find $x \in \mathbb{R}$ such that the angle between $a$ and $b$ is $\pi/4$.

Problem 2. [Gunnarsson, Spring 2007, 2] Consider the two vectors $a = <1, 2, 3>$ and $b = <2, 3, 4>$. Calculate the following:
(a) The cosine of the angle between $a$ and $b$;
(b) $a \times b$;
(c) The area of the parallelogram with corner points $P(0, 0, 0)$, $Q(1, 2, 3)$, $R(2, 3, 4)$, $S(3, 5, 7)$.

Problem 3. [Loveless, Winter 2007, 1bc] Let $a = <3, -1, 2>$ and $b = 5i - 7j + 2k$.
(b) Find $a \cdot b$.
(c) Find the angle, $\theta$, between the vectors $a$ and $b$. Give your answer in radians such that $0 \leq \theta \leq \pi$.
(Round your answer to 3 digits after the decimal point.)

Problem 4. [Loveless, Winter 2007, 2] (a) Find all values of $x$ so that $a = <1, x, -4>$ and $b = <x, 3, 5>$ are orthogonal.
(b) Find the unit vector that is orthogonal to both $a = <1, 4, 5>$ and $b = <-1, 3, 0>$.

Problem 5. [Conroy, Winter 2006, 5] Suppose the vector $<x, 3, 2>$ is orthogonal to the vector $<2, 3, x>$. Find $x$.

Note: these problems are taken from old first midterms of Math 126. These old midterms can be found at http://www.math.washington.edu/m126/midterms/midterm1.php

Solutions

Problem 1. (a) $a$ and $b$ are orthogonal if and only if the angle $\theta$ between them is $\pi/2$, i.e. if and only if $\cos \theta = 0$. But
\[
\cos \theta = \frac{a \cdot b}{|a||b|}.
\]
Hence $\cos \theta = 0$ if and only if $a \cdot b = 0$. In other words, the two vectors $a, b$ are orthogonal if and only if $a \cdot b = 0$. But $a \cdot b = 1 \cdot 0 + 1 \cdot x + x \cdot 1 = 2x$. Hence we have: $2x = 0$, $x = 0$. The vectors $a, b$ are orthogonal if and only if $x = 0$.

(b) $\theta = \pi/4$ if and only if $\cos \theta = \sqrt{2}/2$. We have:
\[
\frac{\sqrt{2}}{2} = \frac{a \cdot b}{|a||b|}.
\]
But $a \cdot b = 2x$ (see above), $|a| = \sqrt{1^2 + 1^2 + x^2} = \sqrt{2 + x^2}$, $|b| = \sqrt{0^2 + x^2 + 1^2} = \sqrt{x^2 + 1}$. Hence
\[
\frac{\sqrt{2}}{2} = \frac{2x}{\sqrt{x^2 + 2\sqrt{x^2 + 1}}},
\]
\[
\frac{1}{2} = \frac{4x^2}{(x^2 + 2)(x^2 + 1)},
\]
\[
8x^2 = (x^2 + 2)(x^2 + 1), \quad 8x^2 = x^4 + 3x^2 + 2,
\]
\[ x^4 - 5x^2 + 2 = 0, \quad x^2 = \frac{5 \pm \sqrt{17}}{10} \]

\[ x = \pm \sqrt{\frac{5 \pm \sqrt{17}}{10}}. \]

(Because \( \frac{5 \pm \sqrt{17}}{10} > 0 \).)

**Problem 2.** (a) \( \mathbf{a} \cdot \mathbf{b} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 = 20 \), \( |\mathbf{a}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14} \), \( |\mathbf{b}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29} \).

Thus, if \( \theta \) is the angle between \( \mathbf{a} \) and \( \mathbf{b} \), we have:

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{20}{\sqrt{14}\sqrt{29}}.
\]

(b) \( \mathbf{a} \times \mathbf{b} = \langle 2 \cdot 4 - 3 \cdot 3, 3 \cdot 2 - 4 \cdot 1, 1 \cdot 3 - 2 \cdot 2 \rangle = \langle -1, 2, -1 \rangle \).

(c) This area is \( |\mathbf{PQ} \times \mathbf{PR}| = |\mathbf{a} \times \mathbf{b}| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6} \).

**Problem 3.** (a) \( \mathbf{b} = \langle 5, -7, 2 \rangle \). Hence \( \mathbf{a} \cdot \mathbf{b} = 3 \cdot 5 + (-1) \cdot (-7) + 2 \cdot 2 = 26 \).

(b) \( |\mathbf{a}| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14} \), \( |\mathbf{b}| = \sqrt{5^2 + (-7)^2 + 2^2} = \sqrt{78} \). Thus

\[
\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{26}{\sqrt{14}\sqrt{78}}.
\]

\[ \theta \approx 38.113^\circ \approx 0.665. \]

(The last number is the radian measure of \( \theta \). You may write any angle either in degrees or radians, but the radians are preferable.)

**Problem 4.** (a) As in Problem 1a, \( \mathbf{a} \) and \( \mathbf{b} \) are orthogonal if and only if \( \mathbf{a} \cdot \mathbf{b} = 0 \). But \( \mathbf{a} \cdot \mathbf{b} = 1 \cdot x + x \cdot 3 + (-4) \cdot 5 = 4x - 20 \). Hence we have: \( 4x - 20 = 0, x = 5 \).

(b) \( \mathbf{c} := \mathbf{a} \times \mathbf{b} = \langle -15, -5, 7 \rangle \) is orthogonal to both \( \mathbf{a} \) and \( \mathbf{b} \). But it is not a unit vector; we have to normalize it: \( |\mathbf{c}| = \sqrt{(-15)^2 + (-5)^2 + 7^2} = \sqrt{299} \), and the vector

\[
\mathbf{n} = \frac{\mathbf{c}}{|\mathbf{c}|} = \langle -\frac{15}{\sqrt{299}}, -\frac{5}{\sqrt{299}}, \frac{7}{\sqrt{299}} \rangle
\]

is a unit vector orthogonal to \( \mathbf{a} \) and \( \mathbf{b} \).

**Problem 5.** Similarly to Problems 1a and 4a, \( \langle x, 3, 2 \rangle \cdot \langle 2, 3, x \rangle = x \cdot 2 + 3 \cdot 3 + 2 \cdot x = 4x + 9 \), and these vectors are orthogonal if and only if \( 4x + 9 = 0, x = -9/4 \).