Problem 1. [Pevtsova, Win07, 2] Let \( \mathbf{r}(t) = \langle t, t^2, t^3 \rangle \). Find the equation of the normal plane:
(a) at \( t = 1 \);
(b) at \( \mathbf{r}(t) = \langle -1, 1, -1 \rangle \);
(c) find the parametric equations of the line of intersection of these two normal planes.

Problem 2. [Pevtsova, Win07, 4] (a) Find the velocity and position vectors \( \mathbf{v}(t), \mathbf{r}(t) \) of a particle if \( \mathbf{a}(t) = \langle 2, \cos t, \sin t \rangle \), and at the moment \( t = 0 \) we have \( \mathbf{v}(0) = \langle 0, 0, -1 \rangle, \mathbf{r}(0) = \langle 1, 1, 0 \rangle \).
(b) Find the curvature of \( \mathbf{r}(t) \) at \( t = 1 \).

Problem 3. [Milakis, Sp07, 3] (a) Find an equation of the plane through the points \((1, 0, 0), (0, 1, 0), (0, 0, 1)\).
(b) Compute the distance from the point \( \mathbf{P}(2, 0, 0) \) to the plane described in part (a).

Problem 4. [Milakis, Sp07, 5] Let \( \mathbf{r}(t) = \langle t^2, 2t, t \rangle \).
(a) Find the unit tangent vector \( \mathbf{T} \), the unit normal vector \( \mathbf{N} \) and the binormal vector \( \mathbf{B} \) at the point where \( \mathbf{T} \) is parallel to the plane \( x + y + z = 0 \).
(b) Find the curvature of \( \mathbf{r}(t) \) at the point identified in part (a).

Problem 5. [Pevtsova, Win07, 5] Show that if a particle moves with a constant speed, then the velocity and acceleration vectors are orthogonal.

Note: these problems are taken from old second midterms of Math 126. These old midterms can be found at http://www.math.washington.edu/ m126/midterms/midterm2.php

Solutions

Problem 1. A normal plane at any point of a curve is the plane which passes through this point and has a tangent vector to this curve as its normal vector. But \( \mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle \). Hence:
(a) at the point where \( t = 1 \) (this point is \((1, 1, 1)\)) we have \( \mathbf{r}'(1) = (1, 2, 3) \), and the equation of the plane is
\[
1 \cdot (x - 1) + 2 \cdot (y - 1) + 3 \cdot (z - 1) = 0, \quad x + 2y + 3z = 6.
\]
(b) at the point \((-1, 1, -1)\) (where \( t = -1 \)) we have \( \mathbf{r}'(1) = (1, -2, 3) \), and the equation of the plane is
\[
1 \cdot (x + 1) + (-2) \cdot (y - 1) + 3 \cdot (z + 1) = 0, \quad x - 2y + 3z + 6 = 0.
\]
(c) take, e.g. \( z = t \) and obtain the system of equations (\( t \) is not considered as an unknown variable, it is a parameter):
\[
x + 2y = 6 - 3t, \quad x - 2y = -6 - 3t.
\]
Add these two equations: \( 2x = -6t, x = -3t \). Subtract the second from the first: \( 4y = 12, y = 3 \). Hence the parametric equations of this line of intersection are:
\[
x = -3t, y = 3, z = t.
\]

Problem 2. (a) Use the fundamental theorem of calculus for the vector function \( \mathbf{r}'(t) \):
\[
\mathbf{r}'(t) = \int_0^t \mathbf{r}''(s) ds + \mathbf{r}'(0) = \int_0^t (2i + \cos s j + \sin sk) ds - \mathbf{k} = i \int_0^t 2ds + j \int_0^t \cos s ds + k \int_0^t \sin s ds - \mathbf{k} =
\]
\[
= 2t\mathbf{i} + \sin t\mathbf{j} + (1 - \cos t)\mathbf{k} - \mathbf{k} = 2t\mathbf{i} + \sin t\mathbf{j} - \cos t\mathbf{k}.
\]

Similarly,
\[
\mathbf{r}(t) = \int_0^t \mathbf{r}'(s)ds + \mathbf{r}(0) = \int_0^t (2s\mathbf{i} + \sin s\mathbf{j} - \cos s\mathbf{k})ds + \mathbf{i} + \mathbf{j} = \mathbf{i}\int_0^t 2ds + \mathbf{j}\int_0^t \sin ds - \mathbf{k}\int_0^t \cos ds + \mathbf{i} + \mathbf{j} = \mathbf{i}s^2\big|_{s=0}^{s=t} + \mathbf{j}(-\cos s)\big|_{s=0}^{s=t} - \mathbf{k}\sin s\big|_{s=0}^{s=t} + \mathbf{i} + \mathbf{j} = (t^2 + 1)\mathbf{i} + (2 - \cos t)\mathbf{j} - \sin t\mathbf{k}.
\]

(b) The curvature:
\[
\mathbf{r}'(1) = 2\mathbf{i} + \sin 1\mathbf{j} - \cos 1\mathbf{k} = < 2, \sin 1, -\cos 1 >, \mathbf{r}''(1) = < 2, \cos 1, \sin 1 >, \\
\mathbf{r}'(1) \times \mathbf{r}''(1) = \mathbf{i} - 2(\cos 1 + \sin 1)\mathbf{j} + 2(\cos 1 - \sin 1)\mathbf{k}, \\
|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{1^2 + 4(\cos 1 + \sin 1)^2 + 4(\cos 1 - \sin 1)^2} = \\
= \sqrt{1 + 4(\cos^2 1 + 2\sin 1 \cos 1 + \sin^2 1) + 4(\cos^2 1 - 2\sin 1 \cos 1 + \sin^2 1)} = \\
= \sqrt{1 + 4 + 8\sin 1 \cos 1 + 4 - 8\sin 1 \cos 1} = \sqrt{9} = 3, \\
|\mathbf{r}'(1)| = \sqrt{2^2 + \sin^2 1 + \cos^2 1} = \sqrt{4 + 1} = \sqrt{5},
\]

and the curvature at the point \( t = 1 \) is equal to
\[
\frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^2} = \frac{3}{\sqrt{5}^2} = \frac{3\sqrt{5}}{25}.
\]

**Problem 3.** (a) This plane contains vectors
\[
(1, 0, 0) - (0, 1, 0) = < 1, -1, 0 >
\]
and
\[
(1, 0, 0) - (0, 0, 1) = < 1, 0, -1 >.
\]
Hence their cross product is a normal vector:
\[
\mathbf{n} = < 1, -1, 0 > \times < 1, 0, -1 > = < 1, 1, 1 >.
\]
This plane contains the point \((1, 0, 0)\), hence its equation is
\[
(x - 1) \cdot 1 + (y - 0) \cdot 1 + (z - 0) \cdot 1 = 0, \quad x + y + z - 1 = 0.
\]
(b) The distance from the point \((2, 0, 0)\) to this plane is
\[
\frac{|2 + 0 + 0 - 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}.
\]

**Problem 4.** (a) We shall use formulas from the end of Section 13.3, Stewart. First of all, \(\mathbf{r}'(t) = < 2t, 2, 1 >\). Hence \(|\mathbf{r}'(t)| = \sqrt{(2t)^2 + 2^2 + 1^2} = \sqrt{4t^2 + 5},
\]
\[
\mathbf{T}(t) = \frac{\mathbf{r}(t)}{|\mathbf{r}'(t)|} = < \frac{2t}{\sqrt{4t^2 + 5}}, \frac{2}{\sqrt{4t^2 + 5}}, \frac{1}{\sqrt{4t^2 + 5}} >
\]
This vector is parallel to \(\mathbf{r}\). Hence it is parallel to the plane \(x + y + z = 0\) if and only if the vector \(\mathbf{r}\) is parallel to this plane. And this condition is equivalent to the following: \(\mathbf{r}\) is orthogonal to the vector
<1,1,1>, which is normal to this plane. These vectors are orthogonal if and only if their dot product equals 0, i.e.
\[ 2t \cdot 1 + 2 \cdot 1 + 1 \cdot 1 = 0, \quad 2t + 3 = 0, \quad t = -3/2. \]

Let us compute the normal vector:
\[
\frac{d}{dt} \left( \frac{2t}{\sqrt{4t^2 + 5}} \right) = \frac{(2t)'\sqrt{4t^2 + 5} - 2t(\sqrt{4t^2 + 5})'}{4t^2 + 5} = \frac{2\sqrt{4t^2 + 5} - 2t \cdot \frac{8t}{2\sqrt{4t^2 + 5}}}{4t^2 + 5} = \frac{2(4t^2 + 5) - 8t^2}{(4t^2 + 5)^{3/2}} = \frac{10}{(4t^2 + 5)^{3/2}}.
\]

Hence, differentiating each component, we obtain:
\[
T'(t) = \left< \frac{10}{(4t^2 + 5)^{3/2}}, -\frac{8t}{(4t^2 + 5)^{3/2}}, -\frac{4t}{(4t^2 + 5)^{3/2}} \right>.
\]

At the point \( t = -3/2 \), we have \( 4t^2 + 5 = 4(-3/2)^2 + 5 = 14 \), and
\[
T(-3/2) = \left< \frac{-3}{14}, \frac{2}{14}, \frac{1}{14} \right> = \frac{1}{\sqrt{14}} < -3, 2, 1 >,
\]
\[
T'(-3/2) = \left< \frac{10}{14^{3/2}}, \frac{12}{14^{3/2}}, \frac{6}{14^{3/2}} \right> = 14^{-3/2} < 10, 12, 6 >.
\]

Hence
\[
N(-3/2) = \frac{T'(-3/2)}{|T'(-3/2)|} = \frac{<10,12,6>}{\sqrt{10^2 + 12^2 + 6^2}} = \frac{1}{\sqrt{280}} <10,12,6> = \frac{1}{\sqrt{70}} <5,6,3>.
\]

And
\[
B(-3/2) = T(-3/2) \times N(-3/2) = \frac{1}{14\sqrt{70}} < -3, 2, 1 > \times <5,6,3> = \frac{1}{14\sqrt{5}} <0,14,-28> = \frac{1}{\sqrt{5}} <0,1,-2>.
\]

(b) The curvature is equal to
\[
\frac{|T'(-3/2)|}{|r'(-3/2)|} = \frac{14^{-3/2}\sqrt{10^2 + 12^2 + 6^2}}{\sqrt{4(-3/2)^2 + 5}} = \frac{14^{-3/2}\sqrt{280}}{\sqrt{14}} = \frac{\sqrt{10}}{14^{3/2}} = \frac{1}{14} \sqrt{\frac{5}{14}}.
\]

**Problem 5.** Let \( v(t) \) be the velocity, \( a(t) = v'(t) \) be the acceleration. If \( |v| = \text{const} \), then \( v \cdot v = |v|^2 = \text{const} \), and differentiating this scalar product, we obtain (thanks to Theorem 3 from Section 13.2, Stewart) that
\[
a \cdot v + v \cdot a = 0, \quad 2a \cdot v = 0, \quad a \cdot v = 0,
\]
hence \( v \) and \( a \) are orthogonal.