Problem 1. [Gunnarsson, Sp07, 5] Let 

\[ z = f(x, y) = \sqrt{e^x / y}. \]

(a) Find the domain of \( f \).  
(b) Calculate \( f_x, f_y \), and \( f_{xy} \).  
(c) Sketch the level curves for \( z = 1 \) and \( z = 2 \).

Problem 2. [Pevtsova, Win07, 3] Consider the surface given by the equation 

\[ f(x, y) = x^2 y + y^3 + x. \]

(a) Find the tangent plane to the surface at the point \((-2, 1, 3)\).  
(b) Find all second partial derivatives of \( f \).

Problem 3. [Arms, Aut06, 4] For the function 

\[ f(x, y) = x^2 \sin(\pi y): \]

(a) Compute \( f_x, f_y, \) and \( f_{xy} \).  
(b) Find the equation of the tangent plane to the graph of \( f \) at the point where \((x, y) = (3, 1)\).  
(c) Find the equations of the line through \((3, 1, f(3, 1))\) and perpendicular to the tangent plane in part (b).

Problem 4. [Perkins, Win09, 4] You wish to build a rectangular box with no top with volume 6 ft\(^3\). The material for the bottom is metal and costs $3.00 a square foot. The sides are wooden and cost $2.00 a square foot. Calculate the dimensions of the box with minimum cost. Use the Second Derivative test to verify that your answer is indeed a minimum.

Problem 5. [Milakis, Win09, 3] Find the tangent plane to the surface given by the graph of 

\[ f(x, y) = \sqrt{28 - 2x^2 - y^2} \]

at \((2, 2)\). Use the linear approximation to estimate \( f(1.95, 2.01) \)

Problem 6. [Milakis, Win09, 4] Find (if any) the absolute maximum and minimum values of 

\[ f(x, y) = 3xy^2 \]

in \( D = \{(x, y) : x \geq 0, y \geq 0, x^2 + y^2 \leq 9\} \).

Solutions

Problem 1.  
(a) \( e^x / y \geq 0 \iff y > 0 \), since for all \( x \) \( e^x > 0 \); so the domain is \( \{(x, y) : y > 0\} \).  
(b) Note that \( f = e^{x/2}y^{-1/2} \). Hence \( f_x = (e^{x/2})'y^{-1/2} = (1/2)e^{x/2}y^{-1/2} \),  
\( f_y = e^{x/2}(y^{-1/2})' = e^{x/2}(-1/2)y^{-3/2} = -(1/2)e^{x/2}y^{-3/2} \),  
\( f_{xy} = (1/2)e^{x/2}y^{-1/2}y = 1/2e^{x/2}(y^{-1/2})_y = (1/2)e^{x/2}(-1/2)y^{-3/2} = -(1/4)e^{x/2}y^{-3/2} \).  
(c) \( z = 1 \iff y = e^x \), \( z = 2 \iff y = e^{x/4} \).  

Problem 2.  
(a) Let us find \( f_x(-2, 1), f_y(-2, 1) \). \( f_x = 2xy + 1 \), hence \( f_x(-2, 1) = -3 \). \( f_y = x^2 + 3y^2 \), hence \( f_y(-2, 1) = 7 \). Thus, the equation of this plane is  
\[ z - 3 = -3(x + 2) + 7(y - 1), \quad 3x - 7y + z + 10 = 0. \]
(b) \( f_{xx} = 2y, \ f_{xy} = f_{yx} = 2x, \ f_{yy} = 6y. \)

**Problem 3.**

(a) \( f_x = 2x \sin(\pi y), \ f_y = \pi x^2 \cos(\pi y), \ f_{xy} = 2\pi x \cos(\pi y). \)

(b) First, note that \( f(3,1) = 0. \) Let us find \( f_x(3,1), f_y(3,1). \) \( f_x(3,1) = 2 \cdot 3 \sin(\pi) = 0, \ f_y(3,1) = \pi 3^2 \cos(\pi) = -9\pi. \) Thus, the equation of this plane is

\[
z - 0 = -0(x - 3) + (-9\pi)(y - 1), \quad z + 9\pi y = 9\pi.
\]

(c) Any directional vector of this line is a normal vector to the tangent plane at the point \((x, y) = (3, 1).\) E.g. we can take \(<0, 9\pi, 1>\) (indeed, look at the equation of this plane). Since this line has the point \((3, 1, f(3,1)) = (3, 1, 0),\) the equation of the line is \(x = 0t + 3 = 3, \ y = 9\pi t + 1, \ z = t. \) (Of course, there may be equivalent equations, i.e. different answers that are still correct.)

**Problem 4.** Suppose the length is \( x \) ft, the width is \( y \) ft, the height is \( z \) ft. Then the volume is \( xyz \) ft\(^3\), hence \( xyz = 6. \) The area of the bottom is \( xyft^2, \) hence its cost is \( 3xy \$. \) The total area of the sides is \( 2xz + 2yz \) ft\(^2\) (since there are fours sides, two have area \( xz ft^2 \) and two have area \( yz ft^2 \)). Hence its cost is \( 2(2xz + 2yz) \$. \) From now on, we will eliminate the dollar sign for the sake of brevity. The total cost is

\[
f := 3xy + 4xz + 4yz.
\]

But \( z = 6/(xy), \) hence \( f \) can be expressed as a function of two variables \( x, y: \)

\[
f(x, y) = 3xy + \frac{24}{x} + \frac{24}{y}.
\]

Let us find its critical points:

\[
f_x = 3y - \frac{24}{x^2} = 0, \quad f_y = 3x - \frac{24}{y^2} = 0.
\]

We need to solve this system of equations. After simple algebraic operations, we obtain:

\[
x^2y = 8, \quad xy^2 = 8.
\]

Divide the first equation by the second; obtain: \( x/y = 1, \ x = y. \) Hence \( x^3 = 8, \ x = 2, \ y = 2. \) You do not need to verify that this is indeed a maximal point, on the midterm, unless it is required explicitly (as in this case).

\[
f_{xx} = \frac{48}{y^3}, \quad f_{xy} = f_{yx} = 3, \quad f_{yy} = \frac{48}{x^3}.
\]

Plug in \( x = y = 2: \)

\[
f_{xx}(2,2) = \frac{48}{8} = 6, \quad f_{xy}(2,2) = f_{yx}(2,2) = 3, \quad f_{yy}(2,2) = \frac{48}{8} = 6.
\]

Now we apply the Second Derivative Test: since \( f_{xx}(2,2) > 0 \) and \( f_{xx}(2,2)f_{yy}(2,2) - f_{xy}(2,2)^2 = 6 \cdot 6 - 3^2 = 27 > 0, \) we see: \( (2,2) \) is indeed a local minimum point. And \( z = 6/(xy) = 6/4 = 3/2. \)

The answer: The length and the width of the bottom are \( 2 \) ft each, the height is \( 1.5 = 3/2 \) ft.

**Problem 5.** \( f(2,2) = \sqrt{28 - 8 - 4} = \sqrt{16} = 4. \)

\[
f_x = \frac{(28 - 2x^2 - y^2)}{2\sqrt{28 - 2x^2 - y^2}} = \frac{-4x}{2\sqrt{28 - 2x^2 - y^2}} = \frac{-2x}{\sqrt{28 - 2x^2 - y^2}};
\]

\[
f_y = \frac{(28 - 2x^2 - y^2)}{2\sqrt{28 - 2x^2 - y^2}} = \frac{-2y}{2\sqrt{28 - 2x^2 - y^2}} = \frac{-y}{\sqrt{28 - 2x^2 - y^2}}.
\]
Plug in $x = y = 2$: $\sqrt{28 - 2x^2 - y^2} = 4$, hence

$$fx(2, 2) = \frac{-4}{4} = -1, \quad fy(2, 2) = \frac{-2}{4} = -\frac{1}{2}.$$  

Thus, the tangent plane is

$$z - f(2, 2) = fx(2, 2)(x - 2) + fy(2, 2)(y - 2),$$

or, in other words

$$z - 4 = (-1)(x - 2) + \left(-\frac{1}{2}\right)(y - 2) = -x - \frac{y}{2} + 3, \quad z = -x - \frac{y}{2} + 7.$$  

This equation can be considered as a linear approximation of the function $f$ in the neighborhood of $(2, 2)$. For example, $f(1.95, 2.01) \approx -1.95 - 2.01/2 + 7 = -2 + 0.05 - 1 - 0.005 + 7 = 4 + 0.045 = 4.045$.

**Problem 6.** It is obvious that $f(x, y) \geq 0$ for any point $(x, y) \in D$. And $f = 0 \Leftrightarrow x = 0$ or $y = 0$. Hence $f_{\min} = 0$, attained at any point with $x = 0$ or $y = 0$ on the boundary.

It is much more difficult to find $f_{\max}$. First, let us find the critical points inside $D$:

$$fx = 3y^2 = 0, \quad fy = 6xy = 0;$$

but this implies $y = 0$, i.e. there is no critical point inside $D$ (only on the boundary, and we cannot take them into account if we try to find the maximal value in $D$). Hence $f$ does not attain its maximal value in $D$ in the interior of $D$. It attains this maximal value in $D$ on the boundary of $D$. But what is this value? The boundary consists of three pieces:

$$x = 0, \quad y \in [0, 3]; \quad x \in [0, 3], \quad y = 0; \quad x^2 + y^2 = 9, \quad x, y \geq 0 \Rightarrow x, y \leq 3.$$  

On the first and second pieces we have $f = 0$. And on the third piece

$$f = 3x(9 - x^2) = 27x - 3x^3, \quad x \in [0, 3].$$

Let us find its maximal value on this interval (now we temporarily consider $f$ as a function of one variable).

$$fx = 27 - 9x^2 = 0, \quad x^2 = 3, \quad x = \sqrt{3}.$$  

And $f_{xx} = -18x < 0$ for $x = \sqrt{3}$. Hence the Second Derivative Test shows that $\sqrt{3}$ is a point of local maximum.

Since this is the only critical point, it is the point of global maximum on $[0, 3]$. Thus $(x, y) = (\sqrt{3}, \sqrt{6})$ (we find the corresponding value of $y$ in this way: $y = \sqrt{9 - x^2} = \sqrt{9 - 3} = \sqrt{6}$) is a maximal point of $f$ at the third piece of the boundary.

We conclude that this is the point of global maximum of $f$ on $D$, since its values on the other two pieces of the boundary are $0 < f(\sqrt{3}, \sqrt{6}) = 18\sqrt{3}$. Thus $18\sqrt{3} = f_{\max}$. Summarizing these results, we have:

$$f_{\min} = 0, \text{ attained at any point } x = 0 \text{ or } y = 0.$$  

$$f_{\max} = 18\sqrt{3}, \text{ attained at } (\sqrt{3}, \sqrt{6}).$$