Challenge Problems  
Math126C. 02/17/2011

Problem 1. [Milakis, Win09, 5] Let

\[ D := \{(x, y) \mid 1 \leq x \leq 2, \ln x \leq y \leq e^x\} \]

Compute the area of \( D \).

Problem 2. [Perkins, Win09, 3a] Evaluate the following integral:

\[
\iint_R xy \sin(x^2y) \, dx \, dy, \quad R := [0, 1] \times [0, \pi/2];
\]

Problem 3. [Perkins, Win09, 3b] Evaluate the following integral:

\[
\iint_D y^2e^{xy} \, dx \, dy, \quad D := \{(x, y) \mid 0 \leq y \leq 3, \ 0 \leq x \leq y\}.
\]

Problem 4. [Conroy, Sp07, 3] Suppose a particle is moving in 3-dimensional space so that its position vector is

\[ r(t) = <t, t^2, \frac{1}{t}>. \]

(a) Find the tangential component of the particle’s acceleration vector at time \( t = 1 \).

(b) Find all values of \( t \) at which the particle’s velocity vector is orthogonal to the particle’s acceleration vector.

Problem 5. [Conroy, Sp07, 4] Consider the curve in the \( xy \)-plane defined by the position vector function

\[ r(t) = <t^2 - 3t, t^2 + 2t>. \]

Find the \( t \)-value of the point of maximum curvature on this curve.

Problem 6. [Milakis, Sp07, 4] A curve is given by the equation \( r = 2(1 - \cos \theta) \) in polar coordinates.

(a) Sketch the curve.

(b) Find all the points on the curve where the tangent line is horizontal.

Solutions

Problem 1. The area of \( D \) is

\[
\iint_D 1 \, dx \, dy.
\]

By Fubini’s Theorem, this integral equals

\[
\int_1^2 \left[ \int_{\ln x}^{e^x} 1 \, dy \right] dx = \int_1^2 (e^x - \ln x) \, dx.
\]

But

\[
\int_1^2 e^x \, dx = e^x \bigg|_{x=1}^{x=2} = e^2 - e.
\]
How to compute the integral of $\ln x$? We need to find the antiderivative of this function. One common mistake was that

$$\int_1^2 \ln x \, dx = \left. \frac{1}{x} \right|_{x=1}^{x=2}.$$

But this is not true, since $(1/x)' \neq \ln x$! To find the antiderivative of $\ln x$, we need to integrate by parts:

$$\int \ln x \, dx = \int x' \ln x \, dx = x \ln x - \int x(\ln x)' \, dx = x \ln x - \int \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x.$$

Hence $(x \ln x - x)' = \ln x$, and by the Fundamental Theorem of Calculus

$$\int_1^2 \ln x \, dx = (x \ln x - x)|_{x=1}^{x=2} = (2 \ln 2 - 2) - (\ln 1 - 1) = 2 \ln 2 - 1 = \ln 4 - 1.$$

Hence the area of $D$ is

$$e^2 - e - 2 \ln 2 - 1.$$

**Problem 2.** First, find this one-variable integral:

$$\int_0^1 xy \sin(x^2y) \, dx = \int_0^y \sin t \, \frac{dt}{2},$$

where we have changed the variables: $t = x^2y$, $0 \leq t \leq y$, $dt = 2xy \, dx$. And

$$\int_0^y \sin t \, \frac{dt}{2} = \frac{1}{2} (-\cos t) \bigg|_{t=0}^{t=y} = \frac{1}{2} (1 - \cos y).$$

Hence by Fubini’s Theorem

$$\iint_R xy \sin(x^2y) \, dxdy = \int_0^{\pi/2} \left[ \int_0^1 xy \sin(x^2y) \, dx \right] \, dy = \int_0^{\pi/2} \frac{1}{2} (1 - \cos y) \, dy =$$

$$= \frac{1}{2} \int_0^{\pi/2} dy - \frac{1}{2} \int_0^{\pi/2} \cos y \, dy = \frac{1}{2} \frac{\pi}{2} - \frac{1}{2} (\sin y) \bigg|_{y=0}^{y=\pi/2} = \frac{\pi}{4} - \frac{1}{2}.$$

**Problem 3.** Since $(ye^{xy})_y = y^2e^{xy}$, we have by the Fundamental Theorem of Calculus:

$$\int_0^y y^2 e^{xy} \, dx = ye^{xy} \bigg|_{x=0}^{x=y} = ye^{y^2} - y.$$

Hence by Fubini’s Theorem

$$\iint_D y^2 e^{xy} \, dxdy = \int_0^3 \left[ \int_0^y y^2 e^{xy} \, dx \right] \, dy = \int_0^3 (ye^{y^2} - y) \, dy = \int_0^3 ye^{y^2} \, dy - \int_0^3 y \, dy.$$

To find the first integral, change variables: $t = y^2$, $0 \leq t \leq 9$, $dt = 2y \, dy$.

$$\int_0^3 ye^{y^2} \, dy = \int_0^9 e^t \, \frac{dt}{2} = \frac{1}{2} e^t \bigg|_{t=0}^{t=9} = e^9 - \frac{1}{2}.$$
And the second integral is much easier to compute:

\[ \int_{0}^{3} y \, dy = \frac{y^2}{2} \bigg|_{y=0}^{y=3} = \frac{9}{2}. \]

Thus, the answer is

\[ \frac{1}{2}(e^9 - 1) - \frac{9}{2} = \frac{e^9}{2} - 5. \]

**Problem 4.** (a) \( r'(t) = \langle 1, 2t, -1/t^2 \rangle, \; r''(t) = \langle 0, 2, 2/t^3 \rangle \). Hence \( r'(1) = \langle 1, 2, -1 \rangle, \; r''(1) = \langle 0, 2, 2 \rangle \). By the formula from Section 12.3, Stewart, the vector projection of \( r''(1) \) onto \( r'(1) \) is

\[
\frac{r'(1) \cdot r''(1)}{|r'(1)|^2} r'(1) = \frac{1 \cdot 0 + 2 \cdot 2 + (-1) \cdot 2}{1^2 + 2^2 + (-1)^2} < 1, 2, -1 > = \frac{1}{3} < 1, 2, -1 >,
\]

and the scalar projection is the absolute value of the vector projection, i.e.

\[ \frac{1}{3} \sqrt{1^2 + 2^2 + (-1)^2} = \frac{\sqrt{6}}{3}. \]

(b)

\[
r'(t) \perp r''(t) \iff r'(t) \cdot r''(t) = 0 \iff 1 \cdot 0 + 2t \cdot 2 + \left(-\frac{1}{t^2}\right) \frac{2}{t^3} = 0 \iff 4t - \frac{2}{t^6} = 0 \iff 4t^6 - 2 = 0 \iff t^6 = \frac{1}{2} \iff t = \pm \frac{1}{\sqrt[6]{2}}.
\]

**Problem 5.** In the three-dimensional space,

\[
r(t) = \langle t^2 - 3t, t^2 + 2t, 0 \rangle, \; r'(t) = \langle 2t - 3, 2t + 2, 0 \rangle, \; r''(t) = \langle 2, 2, 0 \rangle.
\]

Hence

\[
r'(t) \times r''(t) = \langle 0, 0, 2(2t - 3) - 2(2t + 2) \rangle = \langle 0, 0, -10 \rangle;
\]

\[
|r'(t)| = \sqrt{(2t - 3)^2 + (2t + 2)^2} = \sqrt{8t^2 - 4t + 13}.
\]

Thus, the curvature at the point \( t \) is

\[
\frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{10}{(8t^2 - 4t + 13)^{3/2}},
\]

and it is maximal if and only if the denominator of this fraction is minimal, i.e. \( 8t^2 - 4t + 13 \) is minimal. But \( 8t^2 - 4t + 13 = 8(t - 1/4)^2 - 1/2 + 13 \) is minimal at the point \( t = 1/4 \). Hence the curvature is maximal at \( t = 1/4 \).

**Problem 6.** (b) Let us find the parametric equations of this curve:

\[
x = r \cos \theta = 2 \cos \theta (1 - \cos \theta), \; y = r \sin \theta = 2 \sin \theta (1 - \cos \theta).
\]

The tangent line is horizontal if and only if the tangent vector \( \langle x'(\theta), y'(\theta) \rangle \) is horizontal, i.e. if and only if \( y'(\theta) = 0 \). But

\[
y'(\theta) = 2 \cos \theta (1 - \cos \theta) + 2 \sin^2 \theta = 2 \cos \theta - 2 \cos^2 \theta + 2(1 - \cos^2 \theta) = 2 + 2u - 4u^2,
\]

where \( u = \cos \theta \). We solve the equation \( 2 + 2u - 4u^2 = 0 \) and find \( u = 1, -1/2 \). But \( \cos \theta = 1 \iff \theta = 0 \), \( \cos \theta = -1/2 \iff \theta = \pm 2\pi/3 \). In the former case \( r = 0 \), so this is the origin. In the latter case \( r = 2(1 - (-1/2)) = 3 \). Thus, the answer: \( r = 0; \; r = 3, \theta = \pm 2\pi/3 \).