Problem 1. [4 points] Consider the function \( f(x) = (x - 1)^{3/2} \).
(a) Find the second Taylor polynomial \( T_2(x) \) for \( f(x) \) based at \( b = 5 \).
(b) Use \( T_2(x) \) to approximate \( f(5.1) \).

Problem 2. [3 points] Consider the function \( f(x) = (x - 1)^{3/2} \) (the same function as in Problem 1). Use Taylor inequality to find an upper bound for the error in the approximation in 1(b).

Problem 3. [3 points] Find the Taylor series for
\[
\ln \left( \frac{1 + x^2}{1 - x^2} \right)
\]
based at \( b = 0 \).

Solutions

Problem 1. (a)
\[
f(x) = (x - 1)^{3/2} \Rightarrow f(b) = f(5) = 4^{3/2} = 8;
\]
\[
f'(x) = \frac{3}{2}(x - 1)^{1/2} \Rightarrow f'(b) = f'(5) = \frac{3}{2} 4^{1/2} = 3;
\]
\[
f''(x) = \frac{3}{4}(x - 1)^{-1/2} \Rightarrow f''(b) = f''(5) = \frac{3}{4} 4^{-1/2} = \frac{3}{8}.
\]
Thus,
\[
T_2(x) = f(b) + f'(b)(x - b) + \frac{f''(b)}{2}(x - b)^2 = 8 + 3(x - 5) + \frac{3}{16}(x - 5)^2.
\]

(b) \( f(5.1) \approx T_2(5.1) = 8 + 3(5.1 - 5) + \frac{3}{16}(5.1 - 5)^2 = 8.3 + \frac{3}{1600} = 8.301875 \).

Problem 2. The upper bound is
\[
\frac{M}{3!} (5.1 - 5)^3, \quad M := \max_{5 \leq y \leq 5.1} |f'''(y)|.
\]
Let us calculate \( M \). We have:
\[
f'''(y) = \frac{3}{8}(y - 1)^{-3/2},
\]
so
\[
M = \max_{5 \leq y \leq 5.1} \frac{3}{8(y - 1)^{3/2}} = \frac{3}{8(5 - 1)^{3/2}},
\]
because this function decreases and attains its maximum at the left endpoint of the interval \([5, 5.1] \).
But
\[
\frac{3}{8(5 - 1)^{3/2}} = \frac{3}{64}.
\]
Thus, the error bound is
\[
\frac{3/64}{3!} (5.1 - 5)^3 = \frac{1}{128} \cdot 0.1^3 = \frac{1}{128000} = 7.8125 \cdot 10^{-6}.
\]
Problem 3. We know that
\[ \ln(1 + y) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} y^k, \]
so
\[ f(x) = \ln(1 + x^2) - \ln(1 - x^2) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} - 1}{k} x^{2k} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (-x^2)^k. \]
But
\[ \frac{(-1)^{k-1}}{k} (-x^2)^k = \frac{(-1)^{k-1}(-1)^k}{k} x^{2k} = \frac{(-1)^{2k-1}}{k} x^{2k} = -\frac{1}{k} x^{2k}. \]
Thus,
\[ f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} + 1}{k} x^{2k}. \]