Challenge Problems 1
Math126E. 04/05/2011

Problem 1. [Midterm 1, Milakis, Spring 2007, 1] Let \(\mathbf{a} = \langle 1, 1, x \rangle\) and \(\mathbf{b} = \langle 0, x, 1 \rangle\).
(a) Find \(x \in \mathbb{R}\) such that \(\mathbf{a}\) and \(\mathbf{b}\) are orthogonal.
(b) Find \(x \in \mathbb{R}\) such that the angle between \(\mathbf{a}\) and \(\mathbf{b}\) is \(\pi/4\).

Problem 2. [Midterm 1, Gunnarsson, Spring 2007, 2] Consider the two vectors \(\mathbf{a} = \langle 1, 2, 3 \rangle\) and \(\mathbf{b} = \langle 2, 3, 4 \rangle\). Calculate the following:
(a) The cosine of the angle between \(\mathbf{a}\) and \(\mathbf{b}\);
(b) \(\mathbf{a} \times \mathbf{b}\);
(c) The area of the parallelogram with corner points \(P(0, 0, 0), Q(1, 2, 3), R(2, 3, 4), S(3, 5, 7)\).

Problem 3. [Midterm 1, Loveless, Winter 2007, 1bc] Let \(\mathbf{a} = \langle 3, -1, 2 \rangle\) and \(\mathbf{b} = 5\mathbf{i} - 7\mathbf{j} + 2\mathbf{k}\).
(b) Find \(\mathbf{a} \cdot \mathbf{b}\).
(c) Find the angle, \(\theta\), between the vectors \(\mathbf{a}\) and \(\mathbf{b}\). Give your answer in radians such that \(0 \leq \theta \leq \pi\). (Round your answer to 3 digits after the decimal point.)

Problem 4. [Midterm 1, Loveless, Winter 2007, 2] (a) Find all values of \(x\) so that \(\mathbf{a} = \langle 1, x, -4 \rangle\) and \(\mathbf{b} = \langle x, 3, 5 \rangle\) are orthogonal.
(b) Find the unit vector that is orthogonal to both \(\mathbf{a} = \langle 1, 4, 5 \rangle\) and \(\mathbf{b} = \langle -1, 3, 0 \rangle\).

Problem 5. [Midterm 1, Conroy, Winter 2006, 5] Suppose the vector \(\langle x, 3, 2 \rangle\) is orthogonal to the vector \(\langle 2, 3, x \rangle\). Find \(x\).
Solutions

Problem 1. (a) \(a\) and \(b\) are orthogonal if and only if the angle \(\theta\) between them is \(\pi/2\), i.e. if and only if \(\cos \theta = 0\). But
\[
\cos \theta = \frac{a \cdot b}{|a||b|}.
\]
Hence \(\cos \theta = 0\) if and only if \(a \cdot b = 0\). In other words, the two vectors \(a, b\) are orthogonal if and only if \(a \cdot b = 0\). But \(a \cdot b = 1 \cdot 0 + 1 \cdot x + x \cdot 1 = 2x\). Hence we have: \(2x = 0, x = 0\). The vectors \(a, b\) are orthogonal if and only if \(x = 0\).

(b) \(\theta = \pi/4\) if and only if \(\cos \theta = \sqrt{2}/2\). We have:
\[
\frac{\sqrt{2}}{2} = \frac{a \cdot b}{|a||b|}.
\]
But \(a \cdot b = 2x\) (see above), \(|a| = \sqrt{1^2 + 1^2 + x^2} = \sqrt{2 + x^2}, |b| = \sqrt{0^2 + x^2 + 1^2} = \sqrt{x^2 + 1}\). Hence
\[
\frac{\sqrt{2}}{2} = \frac{2x}{\sqrt{2 + 2\sqrt{x^2 + 1}}},
\]
\[
\frac{1}{2} = \frac{4x^2}{(x^2 + 2)(x^2 + 1)},
\]
\[
8x^2 = (x^2 + 2)(x^2 + 1), 8x^2 = x^4 + 3x^2 + 2,
\]
\[
x^4 - 5x^2 + 2 = 0, x^2 = \frac{5 \pm \sqrt{17}}{2}
\]
\[
x = \pm \sqrt{\frac{5 \pm \sqrt{17}}{2}}.
\]
(Because \(\frac{5 \pm \sqrt{17}}{2} > 0\).)

Problem 2. (a) \(a \cdot b = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 = 20, |a| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}, |b| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}\). Thus, if \(\theta\) is the angle between \(a\) and \(b\), we have:
\[
\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{20}{\sqrt{14}\sqrt{29}}.
\]
(b) \(a \times b = \langle 2 \cdot 4 - 3 \cdot 3, 3 \cdot 2 - 4 \cdot 1, 1 \cdot 3 - 2 \cdot 2 \rangle = \langle -1, 2, -1 \rangle\).
(c) This area is \(|PQ \times PR| = |a \times b| = \sqrt{(-1)^2 + 2^2 + (-1)^2} = \sqrt{6}\).

Problem 3. (a) \(b = \langle 5, -7, 2 \rangle\). Hence \(a \cdot b = 3 \cdot 5 + (-1) \cdot (-7) + 2 \cdot 2 = 26\).
(b) \(|a| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}, |b| = \sqrt{5^2 + (-7)^2 + 2^2} = \sqrt{78}\). Thus
\[
\cos \theta = \frac{a \cdot b}{|a||b|} = \frac{26}{\sqrt{14}\sqrt{78}}.
\]
\[
\theta \approx 38.113^\circ \approx 0.665.
\]
(The last number is the radian measure of \(\theta\). You may write any angle either in degrees or radians, but the radians are preferable.)

Problem 4. (a) As in Problem 1a, \(a\) and \(b\) are orthogonal if and only if \(a \cdot b = 0\). But \(a \cdot b = 1 \cdot x + x \cdot 3 + (-4) \cdot 5 = 4x - 20\). Hence we have: \(4x - 20 = 0, x = 5\).
(b) \( \mathbf{c} := \mathbf{a} \times \mathbf{b} = \langle -15, -5, 7 \rangle \) is orthogonal to both \( \mathbf{a} \) and \( \mathbf{b} \). But it is not a unit vector; we have to normalize it: 

\[
|\mathbf{c}| = \sqrt{(-15)^2 + (-5)^2 + 7^2} = \sqrt{299},
\]

and the vector

\[
\mathbf{n} = \frac{\mathbf{c}}{|\mathbf{c}|} = \langle -\frac{15}{\sqrt{299}}, -\frac{5}{\sqrt{299}}, \frac{7}{\sqrt{299}} \rangle
\]

is a unit vector orthogonal to \( \mathbf{a} \) and \( \mathbf{b} \).

**Problem 5.** Similarly to Problems 1a and 4a, 

\[
\langle x, 3, 2 \rangle \cdot \langle 2, 3, x \rangle = x \cdot 2 + 3 \cdot 3 + 2 \cdot x = 4x + 9,
\]

and these vectors are orthogonal if and only if \( 4x + 9 = 0 \), \( x = -9/4 \).