Problem 1. [Midterm 1, Békayl, Spring 2009, 2] Find the angle of intersection of the two curves
\( \mathbf{r}_1(t) = <t^3, 2t^2 + 1, 2t + 3> \) and \( \mathbf{r}_2(s) = <s - 4, s - 3, s - 1> \).

Problem 2. [Midterm 1, Milakis, Winter 2009, 6] Find the exact coordinates of the lowest point on
the curve in \( \mathbb{R}^2 \) given by the parametric equations \( x = 2\cos(t) + \sin(t), \ y = \sin(t) - \cos(t) \).

Problem 3. [Midterm 1, Békayl, Autumn 2007, 1] For the questions below use the points \( P(2, 1, 5), Q(1, 3, 4) \) and \( R(3, 0, 6) \).
(a) Find a vector orthogonal (perpendicular) to the plane through the points \( P, Q \) and \( R \).
(b) Find the area of the triangle \( PQR \).
(c) Determine if the point \( T(0, 3, 3) \) is on the same plane as \( P, Q \) and \( R \).

Problem 4. [Midterm 1, Békayl, Spring 2008, 3] Sketch the graph of the curve
\( x = e^t \cos t, \ y = e^t \sin t, \ 0 \leq t \leq 2\pi \)
marking the \( x \) and \( y \) intercepts and find its length.

Problem 5. [Midterm 1, Pevtsova, Autumn 2006, 4] Let \( A = (3, 0, 0), B = (0, 4, 0), \) and \( C = (0, 0, 1) \).
(a) Find the area of the triangle \( ABC \).
Hint. The following identity may be useful: \( 3^2 + 4^2 + 12^2 = 13^2 \).
(b) Let \( CH \) be the height of the triangle from the vertex \( C \) to the base \( AB \). Find the coordinates of
the point \( H \).
Problem 1. First, let us find the point of intersection of these curves. We must solve the system of equations

\[ t^3 = s - 4, \quad 2t^2 + 1 = s - 3, \quad 2t + 3 = s - 1. \]

Subtract the third equation from the second and obtain \(2t^2 - 2t - 2 = -2, 2t^2 - 2t = 0, 2t(t-1) = 0,\) so \(t = 0\) or \(t = 1.\) Plug in \(t = 0: 0 = s - 4, 1 = s - 3, 3 = s - 1,\) so \(s = 4.\) Plug in \(t = 1: 1 = s - 4, 3 = s - 3, 5 = s - 1,\) so \(s = 5\) and \(s = 6\) - this is impossible, so the case \(t = 1\) does not give us any solution. Thus, the only point of intersection is at \(t = 0, s = 4.\) (This is \(x = 0, y = 1, z = 3.\)

Since \(r_1'(t) = -<3t^2, 4t, 2>,\) the tangent vector to the first curve at this point of intersection is \(a = r_1'(0) = <-0, 0, 2>\). Since \(r_2'(s) = <-1, 1, 1>,\) the tangent vector to the second curve at this point of intersection is \(b = r_2'(4) = <-1, 1, 1>\). The angle between the curves is the angle between \(a\) and \(b.\) If \(\theta\) is this angle, then

\[ \cos \theta = \frac{a \cdot b}{|a||b|}. \]

But \(a \cdot b = 2, |a| = 2, b = \sqrt{3},\) so \(\cos \theta = 1/\sqrt{3},\) and \(\theta = \arccos(1/\sqrt{3}).\)

Problem 2. The lowest point is the point with the minimal \(y.\) Let us find the minimum point of \(y(t) = \sin t - \cos t.\) We can do this without calculus. Indeed,

\[ y(t) = \sqrt{2} \left( \frac{1}{\sqrt{2}} \sin t - \frac{1}{\sqrt{2}} \cos t \right) = \sqrt{2} \left( \cos \frac{\pi}{4} \sin t - \sin \frac{\pi}{4} \cos t \right) = \sqrt{2} \sin \left( t - \frac{\pi}{4} \right). \]

This function attains its (global) minimum when \(t - \pi/4 = -\pi/2, t = -\pi/4.\) Thus, the lowest point is

\[ x(-\pi/4) = 2 \cos(-\pi/4) + \sin(-\pi/4) = 2 \cos(\pi/4) - \sin(\pi/4) = \sqrt{2}/2, \]

\[ y(-\pi/4) = \sin(-\pi/4) - \cos(-\pi/4) = -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2}. \]

Problem 3. (a) The vectors \(a = PQ = <-1, 2, -1>\) and \(b = PR = <-1, -1, 1>\) lie on this plane. Therefore, the following vector is normal: \(n = a \times b = <1, 0, -1>\).

(b) The area of the triangle \(PQR\) is half of the area of the parallelogramm based on the vectors \(a, b.\) The area of this parallelogramm is \(|a \times b| = \sqrt{2}.\) Thus, the area of this triangle is \(\sqrt{2}/2.\)

(c) This plane passes through the point \((2, 1, 5)\) and has a normal vector \(<1, 0, -1>\) Therefore, its equation is

\[ 1 \cdot (x - 2) + 0 \cdot (y - 1) + (-1) \cdot (z - 5) = 0, \quad x - z + 3 = 0. \]

And this point \(T(0, 3, 3)\) satisfies this equation; therefore, it lies on the plane which passes through \(P, Q\) and \(R.\)

Problem 4. Let us find the length of this curve.

\[ x'(t) = e^t (\cos t - \sin t), \quad y'(t) = e^t (\sin t + \cos t). \]

Therefore,

\[ x'(t)^2 + y'(t)^2 = e^{2t} ((\cos t - \sin t)^2 + (\sin t + \cos t)^2) = e^{2t} (\cos^2 t - 2 \cos t \sin t + \sin^2 t + \sin^2 t + \sin^2 t + 2 \cos t \sin t + \cos^2 t) = e^{2t} (2 \cos^2 t + 2 \sin^2 t) = 2e^{2t}. \]

And \(\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{2e^t},\) so the length of this curve is

\[ \int_0^{2\pi} \sqrt{x'(t)^2 + y'(t)^2} dt = \int_0^{2\pi} \sqrt{2} e^t dt = \sqrt{2} (e^{2\pi} - 1). \]
Problem 5. (a) \( \mathbf{a} = \mathbf{AB} = \langle -3, 4, 0 \rangle, \mathbf{b} = \mathbf{AC} = \langle -3, 0, 1 \rangle. \) The area of this triangle (similarly to Problem 3(b)) is

\[
\frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} |\langle 4, 3, 12 \rangle| = \frac{1}{2} \sqrt{4^2 + 3^2 + 12^2} = \frac{13}{2},
\]

because \( \mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 4, 3, 12 \rangle. \)

(b) Let us find the equations of the lines \( \mathbf{CH} \) and \( \mathbf{AB}. \) The directional vector of the line \( \mathbf{AB} \) is \( \mathbf{AB} = \langle -3, 4, 0 \rangle. \) This line passes through the point \( \mathbf{A} = (3, 0, 0). \) Therefore, its parametric equation is

\[
x = 3 - 3t, \quad y = 4t, \quad z = 0.
\]

The line \( \mathbf{CH} \) lies on the plane that passes through the points \( \mathbf{A}, \mathbf{B}, \mathbf{C} \) and therefore is orthogonal to the normal vector \( \mathbf{n} = \langle 4, 3, 12 \rangle. \) This line is also orthogonal to the vector \( \mathbf{AB} = \langle -3, 4, 0 \rangle. \) Therefore, its directional vector is

\[
\mathbf{n} \times \mathbf{AB} = \langle -48, -36, 25 \rangle.
\]

This line passes through the point \( \mathbf{C} = (0, 0, 1); \) therefore, its equation is

\[
x = -48s, \quad y = -36s, \quad z = 25s + 1.
\]

The point \( \mathbf{H} \) is the point of intersection of these lines; to find it, we need to solve the system

\[
-48s = 3 - 3t, \quad -36s = 4t, \quad 25s + 1 = 0.
\]

\( s = -1/25 \) from the third equation, so \( x = 48/25, y = 36/25, z = 0 \) are the coordinates of \( \mathbf{H}. \)