**Problem 1.** [Final Exam, Spring 2007, 5] What is the maximal curvature of the curve \( y = \ln \cos x \)?

**Problem 2.** [Midterm 2, Milakis, Spring 2009, 4] A curve is given by the equation \( r = 2(1 - \cos \theta) \) in polar coordinates.
(a) Sketch the curve.
(b) Find all the points on the curve where the tangent line is horizontal.

**Problem 3.** [Midterm 2, Bekyel, Autumn 2007, 3] Consider the curve
\[ \mathbf{r}(t) = \langle t^2, \cos(t^3), \sin(t^3) \rangle. \]
(a) Find the length of the curve from \( t = 0 \) to \( t = 2\pi \).
(b) Reparametrize the curve with respect to arc length measured from the point \( t = 0 \).

**Problem 4.** [Midterm 2, Pevtsova, Winter 2007, 1] Consider the curve given by the equation in polar coordinates
\[ r = 4 \cos \theta + \sin \theta. \]
(a) Find the Cartesian equation of the curve. Sketch the curve.
(b) Find the equation of the tangent line to the curve at the point \( \theta = \pi/4 \).

**Problem 5.** [Midterm 2, Milakis, Winter 2009, 2] Reparametrize the curve
\[ \left\langle \frac{2}{t^2 + 1} - 1, \frac{2t}{t^2 + 1}, 1 \right\rangle \]
with respect to arc length measured from point \((1, 0, 1)\) in the direction of increasing \( t \). Express the reparametrization in its simplest form.
Solutions

Problem 1. The curve is given by the equation
\[ \mathbf{r}(t) = < t, \ln \cos t, 0 >. \]
Therefore,
\[ \mathbf{r}'(t) = < 1, -\frac{\sin t}{\cos t}, 0 >, \]
\[ \mathbf{r}''(t) = < 0, -\frac{1}{\cos^2 t}, 0 >. \]
Then we get
\[ \mathbf{r}'(t) \times \mathbf{r}''(t) = < 0, 0, -\frac{1}{\cos^2 t} >, \quad | \mathbf{r}'(t) \times \mathbf{r}''(t) | = \frac{1}{\cos^2 t}. \]
Also, \( |\mathbf{r}'(t)| = \sqrt{1 + \tan^2 t} = \sqrt{1/\cos^2 t} = 1/| \cos t | \), and the curvature is
\[ k(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{1/\cos^2 t}{1/| \cos t |^3} = \frac{1/| \cos t |^2}{1/| \cos t |^3} = | \cos t |. \]
The maximum value of this function is 1.

Problem 2. (b) The parametric equations of this curve in Cartesian coordinates are
\[ x = r \cos \theta = 2 \cos \theta (1 - \cos \theta) = 2 \cos \theta - 2 \cos^2 \theta, \]
\[ y = r \sin \theta = 2 \sin \theta (1 - \cos \theta) = 2 \sin \theta - 2 \sin \theta \cos \theta. \]
The directional vector of the tangent line at point \( \theta \) is
\[ < x'(\theta), y'(\theta) > = < -2 \sin \theta + 4 \cos \theta \sin \theta, 2 \cos \theta - 2 \cos^2 \theta + 2 \sin^2 \theta >. \]
This line is horizontal if and only if the \( y \)-component of this directional vector is 0, i.e.
\[ 2 \cos \theta - 2 \cos^2 \theta + 2 \sin^2 \theta = 0. \]
Let us solve this equation. Denote \( u := \cos \theta \) and observe that \( \sin^2 \theta = 1 - u^2 \). So
\[ u - u^2 + (1 - u^2) = 0, \quad 2u^2 - u - 1 = 0, \quad 2(u-1)(u + \frac{1}{2}) = 0, \]
and either \( u = \cos \theta = 1 \) or \( u = \cos \theta = -1/2 \). In the first case, \( \theta = 0 \) and \( x = 0, \ y = 0 \). But in fact, this case is invalid, because \( x'(0) = y'(0) = 0 \), and this means that the tangent line simply does not exist! In the second case, \( \theta = \pm 2\pi/3 \) and \( x = 2u - 2u^2 = -3/2, \ y = \pm (\sqrt{3} + \sqrt{3}/2) = \pm 3\sqrt{3}/2. \) So there are two such points: \((-3/2, 3\sqrt{3}/2), (-3/2, -3\sqrt{3}/2)\).

Problem 3. (a) \( \mathbf{r}'(t) = < 2t, -3t^2 \sin t^3, 3t^2 \cos t^3 >. \) So
\[ |\mathbf{r}'(t)| = \sqrt{4t^2 + (3t^2)^2 \cos^2 t^3 + (3t^2)^2 \sin^2 t^3} = \sqrt{4t^2 + 9t^4} = t \sqrt{4 + 9t^2}. \]
The length from 0 to \( t \) is
\[ s(t) = \int_0^t |\mathbf{r}'(u)|du = \int_0^t u \sqrt{4 + 9u^2}du = \frac{4 + 9t^2}{18} \int_0^4 \sqrt{v} \]
(we changed variables \( v = 4 + 9u^2, \ dv = 18udu \)). Therefore,
\[ s(t) = \frac{1}{18} \left[ \frac{v^{3/2}}{3/2} \right]_4^{4 + 9t^2} = \frac{1}{27} \left( (4 + 9t^2)^{3/2} - 4^{3/2} \right) = \frac{1}{27} \left( (4 + 9t^2)^{3/2} - 8 \right). \]
In particular, \[ s(2\pi) = \frac{1}{27} \left( (4 + 36\pi^2)^{3/2} - 8 \right). \]

(b) We have: \(27s + 8 = (4 + 9t^2)^{3/2}, (27s + 8)^{2/3} - 4 = 9t^2, \) so \(t = \frac{1}{3}((27s + 8)^{2/3} - 4)^{1/2}.\) Thus, \[ \mathbf{r}(t) = < \frac{1}{9}((27s + 8)^{2/3} - 4), \cos \frac{1}{27}((27s + 8)^{2/3} - 4)^{3/2}, \sin \frac{1}{27}((27s + 8)^{2/3} - 4)^{3/2}>. \]

**Problem 4.** (a) \(x = r \cos \theta = 4 \cos^2 \theta + \cos \theta \sin \theta, y = r \sin \theta = 4 \cos \theta \sin \theta + \sin^2 \theta.\) So \(x = 2(\cos 2\theta + 1) + \frac{1}{2} \sin 2\theta, \ y = 2 \sin 2\theta + \frac{1 - \cos 2\theta}{2}.\)

And we have:
\[ x - 2 = 2 \cos 2\theta + \frac{1}{2} \sin 2\theta, \ y - \frac{1}{2} = 2 \sin 2\theta - \frac{1}{2} \cos 2\theta, \]
\[ (x - 2)^2 + (y - \frac{1}{2})^2 = (2 \cos 2\theta + \frac{1}{2} \sin 2\theta)^2 + (2 \sin 2\theta - \frac{1}{2} \cos 2\theta)^2 = \frac{17}{4} \]
(just expand it out and use the trig identity \(\sin^2 \theta + \cos^2 \theta = 1).\)

(b) The coordinates of the point on the curve where \(\theta = \pi/4\) are \(x = 5/2, y = 5/2\) (after plugging in \(\theta = \pi/4\)). Since \(x'(\theta) = -8 \cos \theta \sin \theta + \cos^2 \theta - \sin^2 \theta, \ y'(\theta) = 4(\cos^2 \theta - \sin^2 \theta) + 2 \sin \theta \cos \theta,\) the tangent vector at this point is \(< x'(\pi/4), y'(\pi/4) >= < -4, 1 >.\) This vector is a directional vector for the tangent line; and this line passes through the point \((5/2, 5/2).\) Therefore, \(x = -4t + 5/2, y = t + 5/2\) is the parametric equation of this tangent line.

**Problem 5.**
\[ \mathbf{r}'(t) = < -\frac{4t}{(1 + t^2)^2}, \frac{2(t^2 + 1) - 2t \cdot 2t}{(t^2 + 1)^2}, 0 >= < -\frac{4t}{(t^2 + 1)^2}, \frac{2(1 - t^2)}{(t^2 + 1)^2}, 0 >. \]

\[ |\mathbf{r}'(t)| = \sqrt{\left( -\frac{4t}{(t^2 + 1)^2} \right)^2 + \left( \frac{2(t^2 - 1)}{(t^2 + 1)^2} \right)^2 + 0^2} = \sqrt{\frac{16t^2 + 4(t^2 - 1)^2}{(1 + t^2)^4}} = \sqrt{\frac{4(t^2 + 1)^2}{(t^2 + 1)^4}} = \frac{2}{t^2 + 1}. \]

The point \((1,0,1)\) corresponds to \(t = 0.\) Indeed, compare the second components: \(2t/(t^2 + 1) = 0\) if and only if \(t = 0.\) The arclength from \(t = 0\) to \(t\) is
\[ s(t) = \int_{0}^{t} |\mathbf{r}'(u)|du = \int_{0}^{t} \frac{2}{u^2 + 1}du = 2 \arctan t. \]

So \(t = \tan(s(t)/2).\) And
\[ \mathbf{r}(t) = < \frac{2}{\tan^2(s/2) + 1} - 1, \frac{2 \tan(s/2)}{\tan^2(s/2) + 1}, 1 >= < 2 \cos^2 \frac{s}{2} - 1, 2 \sin \frac{s}{2} \cos \frac{s}{2}, 1 >= < \cos s, \sin s, 1 >. \]