Problem 1. [Final Exam, Spring 2007, 8] Let

\[ f(x, y) = \frac{2x^2 + y^2}{\ln(2x - y)}. \]

(a) Find and sketch the domain of \( f \).
(b) Consider the surface \( z = f(x, y) \). Find the equation of the tangent plane to the surface at a point \((x_0, y_0, z_0)\) with \( x_0 = y_0 = e \).
(c) Using the linear approximation at \((e, e)\) estimate \( f(3, 3) \).

Problem 2. [Midterm 2, Perkins, Winter 2009, 2] Find the equation of the tangent plane of the function

\[ F(x, y) = \frac{3y - 2}{5x + 7} \]

at the point \((1, 1)\).

Problem 3. [Final Exam, Spring 2008, 9] Find three positive numbers \( x, y, \) and \( z \) whose sum is 100 and for which the product

\[ xyz^3 \]

is maximal.

Problem 4. [Final Exam, Autumn 2007, 9] Find the local maximum and minimum values and the saddle points of the function

\[ f(x, y) = x^3 - 12x - 6y + y^2 + 1. \]

Problem 5. [Final Exam, Spring 2007, 9] You wish to build a large swimming pool in the shape of a parallelepiped. It will essentially be an open-top box made of concrete. One side, however, will be made of glass, so that the pool can be observed from below ground. Concrete costs $15 per square meter, and glass costs $100 per square meter. If the volume of the pool must be 1000 cubic meters, what should the dimensions be to minimize the cost of the pool?
Solutions

Problem 1. (a) The domain of \( f \) is given by the conditions
\[
\ln(2x - y) \neq 0 \iff 2x - y > 0, \quad 2x - y \neq 1 \iff y < 2x, \ y \neq 2x - 1.
\]

(b) \( z_0 = f(x_0, y_0) = 3e^2 / \ln e = 3e^2 \).
\[
f_x(x, y) = \frac{4x \ln(2x - y) - (2x^2 + y^2) \frac{2}{2x - y}}{\ln^2(2x - y)},
\]
\[
f_y(x, y) = \frac{2y \ln(2x - y) - (2x^2 + y^2) \frac{-1}{2x - y}}{\ln^2(2x - y)},
\]
and at the point \((e, e)\) we have
\[
f_x(e, e) = \frac{4e \ln e - 3e^2}{\ln^2 e} = -2e, \quad f_y(e, e) = \frac{2e - 3e^2 \frac{1}{e}}{\ln^2 e} = 5e.
\]
So the equation of the tangent plane is
\[
z - z_0 = f_x(e, e)(x - e) + f_y(e, e)(y - e), \quad z - 3e^2 = -2e(x - e) + 5e(y - e) = -2ex + 5ey - 3e^2,
\]
or
\[
z = -2ex + 5ey.
\]
(c) \( f(3, 3) \approx -2e \cdot 3 + 5e \cdot 3 = 9e. \)

Problem 2. \( F(1, 1) = 1/12; \)
\[
F_x(x, y) = -\frac{5(3y - 2)}{(5x + 7)^2}, \quad F_y(x, y) = \frac{3}{5x + 7}.
\]
So \( F_x(1, 1) = -5/144, \ F_y(1, 1) = 1/4. \) The equation of the tangent plane is
\[
z - \frac{1}{12} = -\frac{5}{144}(x - 1) + \frac{1}{4}(y - 1).
\]

Problem 3. Since \( x + y + z = 100, \) we have \( x = 100 - y - z. \) And the expression which we need to maximize is
\[
f(y, z) = (100 - y - z)y^2z^3 = 100y^2z^3 - y^3z^3 - y^2z^4.
\]
Take the derivatives:
\[
f_y = 200yz^3 - 3y^2z^3 - 2y^2z^4, \quad f_z = 300y^2z^2 - 3y^3z^3 - 4y^2z^3.
\]
The system of equations \( f_y = 0, \ f_z = 0 \) can be rewritten as
\[
3y + 2z = 200, \quad 3y + 4z = 300.
\]
Subtract the first equation from the second one and get: \( 2z = 100, \ z = 50; \ 3y = 200 - 2y = 100, \ y = 100/3; \ x = 100 - y - z = 100/6. \) The answer: \( x = 100/6, \ y = 100/3, \ z = 50. \)

Problem 4. First, let us solve the system of equations \( f_x = 0, \ f_y = 0. \)
\[
f_x = 3x^2 - 12, \quad f_y = -6 + 2y.
\]
Therefore,
\[
f_x = f_y = 0 \iff x^2 = 4, \ y = 3 \iff x = \pm 2, \ y = 3.
\]
Also, \( f_{xx} = 6x, \ f_{xy} = 0, \ f_{yy} = 2 \), so \( D := f_{xx}f_{yy} - f_{xy}^2 = 12x \). The point \((2, 3)\) satisfies the conditions \( f_{xx} > 0, \ D > 0 \), so this is a local minimum point. The point \((-2, 3)\) satisfies the condition \( D < 0 \), so it is a saddle point.

**Problem 5.** Suppose \( x \) is the width of the glass side, \( y \) is the width of concrete sides, \( z \) is the height. (Everything is measured in meters.) Then \( xyz \) is the volume of the pool, so \( xyz = 1000, \ z = 1000/xy \).

The area of the glass side is \( xz \), its cost is \( 100xz \). The area of each of the adjacent (concrete) sides is \( yz \), so its cost is \( 15yz \). The area of the opposite (concrete) side is \( xz \), so its cost is \( 15xz \). The area of the concrete bottom is \( xy \), its cost is \( 15xy \). The total cost is

\[
115xz + 30yz + 15xy.
\]

We can express it as a function of \( x, y \):

\[
f(x, y) = 115x \frac{1000}{xy} + 30y \frac{1000}{xy} + 15xy = \frac{115000}{y} + \frac{30000}{x} + 15xy.
\]

Let us find its minimum points. Introduce a system of equations:

\[
f_x = -\frac{30000}{x^2} + 15y = 0, \quad f_y = -\frac{115000}{y^2} + 15x = 0,
\]

so \( y = 2000/x^2 \) (from the first equation). Plug it into the second equation:

\[
-\frac{115000}{(2000/x^2)^2} + 15x = 0 \iff \frac{115000}{4000000} x^4 = 15x \iff x^3 = \frac{60000}{115} = \frac{20000}{23},
\]

so

\[
x = \left( \frac{20000}{23} \right)^{1/3}, \quad y = \frac{2000}{x^2} = \frac{2000}{(20000/23)^{1/3}}, \quad z = \frac{1000}{xy} = \frac{1}{2} \left( \frac{20000}{23} \right)^{1/3}.
\]