Partial Derivatives
Mini-Lecture 02/01/2011. Math 126C

1. **Primary definitions.** From the single-variable calculus, you know that the derivative of a function \( f : \mathbb{R} \to \mathbb{R}, f = f(x), \) at point \( x_0 \) is defined by

\[
f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.
\]

In a similar manner, we define partial derivatives for a function \( f : \mathbb{R}^2 \to \mathbb{R}, f = f(x, y), \) at point \((x_0, y_0)\):

\[
\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}.
\]

In other words, we fix \( y = y_0 \) and vary \( x \), i.e. we consider a function \( g(x) = f(x, y_0) \) and take its derivative at the point \( x = x_0 \): \( g'(x_0) = f_x(x_0, y_0) \). Analogously, we define

\[
\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = \lim_{y \to y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}.
\]

Here, we fix \( x = x_0 \) and vary \( y \), i.e. we consider a function \( h(y) = f(x_0, y) \) and take its derivative at the point \( y = y_0 \): \( h'(y_0) = f_y(x_0, y_0) \).

**Examples.**
1. For \( f(x, y) = 1 \), we have \( f_x = 0 \), \( f_y = 0 \).
2. For \( f(x, y) = xy \), we have \( f_x = y \), \( f_y = x \).
3. For \( f(x, y) = x^2 + y^2 \), we have \( f_x = 2x \), \( f_y = 2y \).

2. **Tangent planes.** You know that the tangent line to the graph of the function \( f(x) \) at the point \( x = x_0, y = y_0 = f(x_0) \) is given by this equation:

\[
y - y_0 = f'(x_0)(x - x_0).\]

Similarly, the tangent plane to the graph \( z = f(x, y) \) of the function \( f(x, y) \) at the point \( x = x_0, y = y_0, z = z_0 = f(x_0, y_0) \) is given by the equation

\[
z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).
\]

**Examples.** Let \( f(x, y) = xy, x_0 = 1, y_0 = 0 \). Then \( z_0 = f(x_0, y_0) = x_0y_0 = 0, f_x(x_0, y_0) = y_0 = 0, f_y(x_0, y_0) = x_0 = 1, \) and the equation of the tangent plane at the point \((x_0, y_0, z_0 = f(x_0, y_0)) = (1, 0, 0)\) is

\[
z - 0 = 0 \cdot (x - 1) + 1 \cdot (y - 0), \quad z = y.
\]

3. **Second-order derivatives.** The second-order derivative of the single-variable function \( f(x) \) is defined as \( f'' = (f')' \), i.e. the derivative of the derivative. In a similar way, let us define second-order partial derivatives: \( f_{xx} = (f_x)_x, f_{xy} = (f_x)_y, f_{yx} = (f_y)_x, f_{yy} = (f_y)_y \). So there are four second-order partial derivatives. (The derivatives \( f_x, f_y \) are called the first-order partial derivatives.)

**Examples.**
1. For \( f(x, y) = xy \), we have: \( f_x = y, f_y = x, f_{xx} = 0, f_{xy} = 1, f_{yx} = 1, f_{yy} = 0 \).
2. For \( f(x, y) = x^2 + y^2 \), we have: \( f_x = 2x, f_y = 2y, f_{xx} = 2, f_{xy} = 0, f_{yx} = 0, f_{yy} = 2 \).

In both examples, you see that \( f_{xy} = f_{yx} \). Indeed, we have the following theorem:

**Clairaut’s Theorem.** If \( f_{xy} \) and \( f_{yx} \) are continuous, then \( f_{xy} = f_{yx} \).

4. **Local maxima and minima.** For one-variable function \( f(x) \), if it has a local maximum or minimum at a point \( x_0 \), then \( f'(x_0) = 0 \). This theorem can be generalized to functions of two variables:
**Theorem.** Suppose \((x_0, y_0)\) is a point of local maximum or minimum of \(f : \mathbb{R}^2 \to \mathbb{R}\), \(f = f(x, y)\). Then \(x_0, y_0\) satisfy the system of equations:

\[
\begin{align*}
    f_x(x_0, y_0) &= 0, \\
    f_y(x_0, y_0) &= 0.
\end{align*}
\] (1)

**Proof.** If we vary two variables and have the local maximum or minimum, and if we fix one of them and vary only the second one, then, of course, we again have the local maximum or minimum. In other words, if we fix \(y = y_0\) and vary \(x\), then \(g(x) = f(x, y_0)\) has local maximum or minimum at the point \(x = x_0\). By the statement from one-variable calculus, \(g'(x_0) = 0\). But recall: \(g'(x_0) = f_x(x_0, y_0)\). Hence \(f_x(x_0, y_0) = 0\). Similarly, if we fix \(x = x_0\) and vary \(y\), then \(h(y) = f(x_0, y)\) has local maximum or minimum at the point \(y = y_0\), and \(f_y(x_0, y_0) = h'(y_0) = 0\). The proof is complete.

That is how we find the candidates for local maximum or minimum points: we solve the system (1).

**Examples.** 1. \(f(x, y) = x^2 + xy + y^2\). Then \(f_x = 2x + y, \ f_y = x + 2y\), and we have the following system:

\[
2x_0 + y_0 = 0, \quad x_0 + 2y_0 = 0.
\]

How to solve this system? \(x_0 = -2y_0, 2(-2y_0) + y_0 = 0, -3y_0 = 0, y_0 = 0, x_0 = 0\). Thus, the only candidate for a point of local maximum or minimum is \((x_0, y_0) = (0, 0)\) - the origin. In fact, it is a local minimum.

2. BUT BEWARE! The solutions of this system need not be the points of local maximum or minimum. For \(f(x, y) = x^2 - y^2\), we have: \(f_x = 2x, f_y = -2y\), so the system of equation is: \(2x_0 = 0, -2y_0 = 0\). Its only solution is \(x_0 = 0, y_0 = 0\). Thus the only candidate for a point of local maximum or minimum is \((0, 0)\) - the origin. But this is neither the point of local maximum nor local minimum. Indeed, take \(x = 0\) and obtain: \(f(x, y) = -y^2 < 0\). Take \(y = 0\) and obtain: \(f(x, y) = x^2 > 0\). So \(f\) can be either larger or smaller than \(f(0, 0) = 0\). Thus, the origin is not a point of local maximum or minimum.

5. **The Second Derivative Test.** Suppose we have a function of one variable, \(f = f(x)\). If \(f'(x_0) = 0\), then we need to check whether \(x_0\) is indeed a point of local maximum or minimum. If \(f''(x_0) < 0\), then \(x_0\) is a point of local maximum. If \(f''(x_0) > 0\), then it is a point of local minimum.

We have a similar test for two variables: if \(f_x(x_0, y_0) = f_y(x_0, y_0) = 0\), then calculate

\[
\Delta = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)
\]

and compare it with zero:
- if \(f_{xx}(x_0, y_0) > 0\) and \(\Delta > 0\), then \((x_0, y_0)\) is the point of local minimum;
- if \(f_{xx}(x_0, y_0) < 0\) and \(\Delta > 0\), then \((x_0, y_0)\) is the point of local maximum.

**Examples.** \(f(x, y) = x^2 + xy + y^2\). (See above.) Then \(f_{xx}(0, 0) = 2 > 0, f_{xy}(0, 0) = 1, f_{yy}(0, 0) = 2\), and \(\Delta = 2 \cdot 2 - 1^2 = 3 > 0\). Thus, \((0, 0)\) is the point of local minimum.