Recall: when we take the derivative of a single-variable function \( y = f(x) \), we say we differentiate with respect to \( x \). This identifies the independent variable in the function and yields the notation \( dy/dx = f'(x) \). For a specific value of \( x \), say \( x = x_0 \), \( f'(x_0) \) gives the slope of the line tangent to the graph of \( y = f(x) \) at the point \((x_0, f(x_0))\). Our goal is to extend the idea of the derivative to multi-variable functions.

Consider the function \( z = f(x, y) = x^2 + 3xy - 2y^2 \). This function has two independent variables. So it has two derivatives: one with respect to \( x \) and the other with respect to \( y \). The partial derivative of \( z = f(x, y) \) with respect to \( x \) is denoted by

\[
\frac{\partial z}{\partial x} = f_x(x, y).
\]

Note. For partial derivatives, we use the notation \( \partial z \), not \( dz \! \)!

You compute this derivative by treating \( y \) as a constant but otherwise differentiating normally:

\[
\frac{\partial z}{\partial x} = f_x(x, y) = 2x + 3y.
\]

Similarly, to compute the partial derivative of \( z = f(x, y) \) with respect to \( y \), treat \( x \) as a constant and \( y \) as a variable:

\[
\frac{\partial z}{\partial y} = f_y(x, y) = 3x - 4y.
\]

Examples.

1. \( f(x, y) = x^2y^3 + \sin(xe^y) \Rightarrow f_x = 2xy^3 + e^y \cos(xe^y), \ f_y = 3x^2y^2 + xe^y \cos(xe^y). \)

2. \( z = x^2e^{3x \cos(xy)} \Rightarrow \frac{\partial z}{\partial x} = x^2e^{3x \cos(xy)} \left(3x(- \sin(xy))y + 3 \cos(xy)\right) + 2xe^{3x \cos(xy)} \),

\[
\frac{\partial z}{\partial y} = x^2e^{3x \cos(xy)} \cdot 3x(- \sin(xy))(x) \cdot.
\]

3. \( g(x, y, z, w) = e^{2xy} - \frac{3z}{w^2 + z} + \log \left(x^2 \sin z + \frac{w}{y}\right) \Rightarrow \)

\[
g_x = e^{2xy}(2y) + \frac{1}{x^2 \sin z + \frac{w}{y}} \cdot (2x \sin z), \ g_y = e^{2xy}2x + \frac{1}{x^2 \sin z + \frac{w}{y}}(- \frac{w}{y^2}),
\]

\[
g_z = -\frac{3(w^2 + z) - 3z}{(w^2 + z)^2} + \frac{1}{x^2 \sin z + \frac{w}{y}} \cdot x^2 \cos z, \ g_w = \frac{3z}{(w^2 + z)^2} \cdot 2w + \frac{1}{x^2 \sin z + \frac{w}{y}} \cdot \frac{1}{y}.
\]